SHORT COMMUNICATION

Differentiation of a Fourier series

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ABSTRACT

It is very known that if the operator $\frac{d}{dx}$ acts on each term into a convergent Fourier series (FS) then it may result a divergent series. This situation is remedied applying the symmetric derivative to FS, which implies the existence of the important Fejér-Lanczos factors. In this note, we show that the orthogonal derivative also leads to these factors.

Keywords: Fourier series, Generalized derivative, Least squares method, Fejér-Lanczos σ-Factors

1. INTRODUCTION

If on the Fourier series [1]:

$$f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} [a_k \cos(kx) + b_k \sin(kx)],$$

(1)

convergent in $[-\pi, \pi]$, we apply the operator $\frac{d}{dx}$ results:
\[
\frac{d}{dx} f(x) = \sum_{k=1}^{\infty} k [-a_k \sin(kx) + b_k \cos(kx)],
\]

which it may be divergent [2, 3]. This problem was remedied by Lanczos [3, 4] with \( f'(x) \)
defined as a Symmetric Derivative [5, 6]:

\[
f'(x) \equiv \lim_{n \to \infty} \frac{1}{\pi n} \left[ f_n \left( x + \frac{\pi}{n} \right) - f_n \left( x - \frac{\pi}{n} \right) \right],
\]

with the partial sums:

\[
f_n(x) = g_n(x) + h_n(x),
\]

\[
g_n(x) = \frac{1}{2} a_0 + \sum_{k=1}^{n} a_k \cos(kx), \quad h_n(x) = \sum_{k=1}^{n} b_k \sin(kx),
\]

resulting the convergent expression:

\[
f'(x) = \lim_{n \to \infty} \sum_{k=1}^{n} \sigma_k \frac{d}{dx} \left[ a_k \cos(kx) + b_k \sin(kx) \right],
\]

with the Fejér-Lanczos Factors [3, 4, 7-10]:

\[
\sigma_0 = 1, \quad \sigma_k = \frac{\sin \left( \frac{k\pi}{n} \right)}{\frac{k\pi}{n}}, \quad k = 1, \ldots, n, \quad \sigma_n = 0;
\]

the set of factors \( \sigma_k \), for a given \( n \), is equivalent to a discrete sampling function. This method
amounts to a multiplication of the standard Fourier coefficients \( a_k \) and \( b_k \) by the ‘attenuation
factors’ \( \sigma_k \).

In (2) and (3) we employ two types of derivatives, however, also there is the orthogonal
it is natural to ask if this ultimate derivative leads to relation (5). The answer is yes, to see the
next Section.

2. THE ORTHOGONAL DERIVATIVE

Lanczos [4, 25] used the least squares method of Legendre [26]-Gauss [27]-Laplace
[28] to obtain an integral expression for the derivative of a function, that is, differentiation by
integration:

\[
F'(x) = \lim_{\epsilon \to 0} \frac{3}{2 \epsilon^3} \int_{-\epsilon}^{\epsilon} t \ F(x + t) \ dt,
\]

which may be applied to Fourier case:

\[
\frac{3}{2 \epsilon^3} \int_{-\epsilon}^{\epsilon} t \ g_n(x + t) \ dt = \frac{3}{2 \epsilon^3} \sum_{k=1}^{n} a_k \int_{-\epsilon}^{\epsilon} t \ \cos(kx + kt) \ dt,
\]
\[= -3 \sum_{k=1}^{n} a_k \frac{\sin(k\epsilon)}{k^2} A_k \quad \text{with} \quad A_k(\epsilon) = \frac{1}{\epsilon} \left[ \sin(k\epsilon) - k \epsilon \cos(k\epsilon) \right] ; \quad (8)\]
similarly:
\[
\frac{3}{2 \epsilon^3} \int_{-\epsilon}^{\epsilon} t h_n(x + t) \, dt = \frac{3}{2 \epsilon^3} \sum_{k=1}^{n} b_k \int_{-\epsilon}^{\epsilon} t \sin(kx + kt) \, dt ,
\]
\[= 3 \sum_{k=1}^{n} b_k \frac{\cos(kx)}{k^2} A_k . \quad (9)\]

Therefore, the Lanczos derivative applied to partial sum (4) gives, taking \( \epsilon = \frac{n}{n} \):
\[
f'(x) = \lim_{n \to \infty} \frac{3}{2 \epsilon^3} \int_{-\epsilon}^{\epsilon} t f_n(x + t) \, dt ,
\]
(8) and (9)
\[= \lim_{n \to \infty} 3 \sum_{k=1}^{n} \frac{1}{k^2} A_k \left[ -a_k \sin(kx) + b_k \cos(kx) \right] ,
\]
\[= \lim_{n \to \infty} \sum_{k=1}^{n} \frac{3A_k}{k^3} \frac{d}{dx} \left[ a_k \cos(kx) + b_k \sin(kx) \right] , \quad (10)\]
but the Bernoulli–Hôpital rule permits to observe the behavior:
\[
A_k \left( \epsilon = \frac{n}{n} \rightarrow n > > 1 \right) \quad \frac{k^3}{3} \frac{\sin(k\epsilon)}{k\epsilon} = \frac{k^3}{3} \frac{\sin(k\frac{\pi}{n})}{k\frac{\pi}{n}} = \frac{k^3}{3} \sigma_k , \quad (11)\]
and this value employed in (10) implies (5), q.e.d.

Thus, it is proved that the symmetric and Cioranescu-(Haslam-Jones)-Lanczos derivatives imply the same expression for the derivative of an infinite Fourier series, with the important participation of the Fejér–Lanczos factors.

### 3. CONCLUSIONS

The orthogonal derivative is a generalization of the symmetric derivative and this is a generalization of the standard derivative. The orthogonal derivative acts as a smoothing (integrating filter):

That is why the highest frequencies of the input are being suppressed in the deduction in Sec. 2, and so this creates also the suppression of the Gibbs phenomenon \([1, 4, 7, 8, 29-32]\). The Legendre polynomials \([1, 4, 33, 34]\) can be employed to extend the method of Cioranescu-(Haslam-Jones)-Lanczos to cover orthogonal derivatives of higher orders \([17, 35, 36]\).
References


