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Using the Moore-Penrose Generalized Inverse in Cryptography

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ABSTRACT

In this work, we introduce a new method in cryptography. It is using the Moore-Penrose generalized inverse of a rectangular matrix to the cryptographic system. We use a rectangular matrix which has the Moore-Penrose generalized inverse as a key. We mean, the rectangular matrix which has full row rank, or the rectangular matrix which has full column rank, or the rectangular matrix which has full factorization.

***Keywords*:** Cryptography, The Moore-Penrose Generalized Inverse, Rectangular Matrix, Plain text, Cipher text, Full Column Rank, Full Row Rank, Full Rank Factorization

1. INTRODUCTION

Many papers studied algebraic methods in cryptography. That is, the algebraic methods which converts a plain message (plain text) into a cipher message (cipher text) [1-9].

In this study, we use the rectangular matrix $R_{m\times n}$, $m \neq n$ which has the Moore-Penrose generalized inverse $R^{\dagger}_{n\times m}$. For that, we must choose the rectangular matrix which is of full row rank (that is, the row rank of this matrix is equal to the number of its rows), or of full column rank (that is, the column rank is equal to the number of its columns), or of full rank factorization. Hence it has the Moore-Penrose generalized inverse.

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It is well known that the 26 letters of the alphabet are given numerical values according to some permutation of the normal sequence $0, 1, 2, \ldots, 25$ such that there is a one-to-one correspondence between the numerical values and the alphabet letters, for example:

Table1. The alphabetic correspondence.

Letter	A	B	C	D	E	F	G	H	I	J	K	L	M	N
Number	0	1	2	3	4	5	6	7	8	9	10	11	12	13

Letter	O	P	Q	R	S	T	U	V	W	X	Y	Z
Number	14	15	16	17	18	19	20	21	22	23	24	25

These numerical values are used for the plain text and the cipher text. That is, any plain text can be converted (enciphered) to a unique cipher text by dividing the plain text into blocks of the letters of the size n or m (because we will use the rectangular matrix $R_{m\times n}$ to obtain the cipher text) and hence applying Table 1. The resulting cipher text can be deciphered to give the plain text using $R^{\dagger}_{n\times m}$.

Throughout this paper, $P_{n\times r}\ (=P)$ denotes the matrix of the numerical values corresponding to the blocks of letters of the plain text, also $P_{r\times m}$. $C_{m\times r}\ (=C)$ denotes the matrix of the numerical values resulting from the encryption, and which converts into letters of the alphabet to obtain the cipher text, also $C_{r\times n}$. R^{T} denotes the transposed of R. I denotes the identity matrix. We will work on the general ring $\mathbb{Z}_{26}$.

This paper is organized as follows: some preliminaries are given in section 2. Section 3 gives the main results. Section 4 gives security analysis, section 5 gives some numerical examples which illustrate our results. Finally section 6 gives conclusions.

2. PRELIMINARIES

In this section, we give important concepts about the Moore-Penrose generalized inverse of a rectangular matrices. For more details see [10-20].

2. 1. Definitions

Definition 2.1.1. (The Moore-Penrose Generalized Inverse of Rectangular matrix)

Let $R_{m\times n}$ be a rectangular matrix, then $R^{\dagger}_{n\times m}$ is unique, and it is called the Moore-Penrose generalized inverse of $R_{m\times n}$ if it satisfies the following conditions:

(1) $R_{m\times n}R^{\dagger}_{n\times m}\,R_{m\times n}=R_{m\times n}$,

(2) $R_{n\times m}^{\dagger} R_{m\times n} R_{n\times m}^{\dagger} = R_{n\times m}^{\dagger}$,

(3) $(R_{m\times n} R_{n\times m}^{\dagger})^{T} = R_{m\times n} R_{n\times m}^{\dagger}$,

(4) $(R_{n\times m}^{\dagger} R_{m\times n})^{T} = R_{n\times m}^{\dagger} R_{m\times n}$.

Definition 2.1.2. [21] (The Full Rank Factorization)

For any m by n matrix R of $rank\ k$, $R = EF$ is said to be a full rank factorization if E and F^{T} have k columns.

Definition 2.1.3. (The Row Echelon Form)

A rectangular m by n matrix B which has $rank\ k$, $R = EF$ is said to be in row echelon form if it is of the form:

$$B = \begin{pmatrix} F_{k\times n} \\ \cdots \\ O_{(m-k)\times n} \end{pmatrix},$$

where O is the zero matrix, and the elements f_{ij} of $F(= F_{k\times n})$ satisfy the following conditions:

(i) $f_{ij} = 0$ when $i > j$.
(ii) The first non-zero entry in each row of F is 1.
(iii) If $f_{ij} = 1$ is the first non-zero entry of the ith row then the jth column of F is the unit vector e_i whose only non-zero entry is in the ith position.

2. 2. Basic Properties of the Moore-Penrose Generalized Inverse

Here, we list some of the more basic properties of the Moore-Penrose generalized inverse of the rectangular matrix R:

(1) $(R^{\dagger})^{\dagger} = R$,

(2) $(R^{\dagger})^{T} = (R^{T})^{\dagger}$,

(3) If λ is a scalar, then $(\lambda R)^{\dagger} = \lambda^{\dagger} R^{\dagger}$ where $\lambda^{\dagger} = \frac{1}{\lambda}$ if $\lambda \neq 0$ and $\lambda^{\dagger} = 0$ if $\lambda = 0$.

(4) $R^{T} = R^{T} R R^{\dagger} = R^{\dagger} R R^{T}$.

(5) $(R^{T} R)^{\dagger} = R^{\dagger} (R^{T})^{\dagger}$.

(6) $R^{\dagger} = (R^{T} R)^{\dagger} R^{T} = R^{T} (R R^{T})^{\dagger}$.

(7) $(URV)^{\dagger} = V^{T} R^{\dagger} U^{T}$, where U, V are unitary matrices.

2. 3. Computation of $R^{\dagger}$

(1) If $R_{m\times n}$, $m = n$ ($R_{m\times n}$ is a square matrix) has full rank, that is $rank(R_{m\times n}) = m = n$ then

$$R^{\dagger}_{n\times m} = R^{-1}_{m\times n}.$$

(2) If $R_{m\times n}$, $m \neq n$ ($R_{m\times n}$ is a rectangular matrix) has full column rank, that is $rank(R_{m\times n}) = n < m$ then $R_{m\times n}$ has left invers

(3)

$$R^{\dagger}_{n\times m} = (R^{T}_{n\times m}R_{m\times n})^{-1}R^{T}_{n\times m} \qquad mod\ M \tag{2.1}$$

where $R^{\dagger}_{n\times m}$ satisfies

$$R^{\dagger}_{n\times m}R_{m\times n} = I \tag{2.2}$$

and

$$R_{m\times n}R^{\dagger}_{n\times m} \neq I$$

(4) If $R_{m\times n}, m \neq n$ ($R_{m\times n}$ is a rectangular matrix) has full row rank, that is $rank(R_{m\times n}) = m < n$ then $R_{m\times n}$ has right invers

$$R^{\dagger}_{n\times m} = R^{T}_{n\times m}(R_{m\times n}R^{T}_{n\times m})^{-1} \quad mod\ M \tag{2.3}$$

where $R^{\dagger}_{n\times m}$ satisfies

$$R_{m\times n}R^{\dagger}_{n\times m} = I \tag{2.4}$$

and

$$R^{\dagger}_{n\times m}R_{m\times n} \neq I$$

(5) **Theorem 2.1. [10].** If $R = EF$ where R is m by n matrix, E is m by k, F is k by n, and $k = rank(R) = rank(E) = rank\ (F)$, then

$$R^{\dagger} = F^{T}(FF^{T})^{-1}\ (E^{T}E)^{-1}\ E^{T}\ . \tag{2.5}$$

Note that, FF^{T} and $E^{T}E$ are k by k matrices of $rank\ k$, so that it makes sense to take their inverses. We can note also

$$R^{\dagger} = \ (EF)^{\dagger} = F^{\dagger}E^{\dagger} \tag{2.6}$$

with

$$F^{\dagger} = F^{T}\ (FF^{T})^{-1} \tag{2.7}$$

a right inverse of F, that is

$$FF^{\dagger} = I \tag{2.8}$$

and

$$E^{\dagger} = (E^T E)^{-1} E^T \tag{2.9}$$

a left inverse of E, that is

$$E^{\dagger} E = I \tag{2.10}$$

(6) Algorithm [10]. To obtain the full rank factorization and the Moore-Penrose generalized inverse for any rectangular matrix $R(= R_{m\times n})$:

(a) Reduce R to row echelon form B_R.
(b) Select the distinguished columns of R (they are the columns which correspond the unit vectors (columns) $e_1, e_2, \dots \dots, e_k$ in B_R) and place them as columns in a matrix E in the same order as they appear in R.
(c) Select the non-zero rows from B_R and place them as rows in a matrix F in the same order as they appear in B_R.
(d) Compute $(FF^T)^{-1}$ and $(E^T E)^{-1}$.
(e) Compute $R^{\dagger}$ as

$$R^{\dagger} = F^T (FF^T)^{-1} (E^T E)^{-1} E^T.$$

2. 4. Proposition

If R is m by n matrix, then there exists E and F are m by k and k by n matrices respectively, such that $R = EF$ and $rank(R) = rank(E) = rank\ (F) = k$.

3. THE MAIN RESULTS

In this section, we will use the rectangular matrix $R_{m\times n}$ where $m \neq n$ in encryption. So, we will use the Moore-Penrose generalized inverse $R^{\dagger}_{n\times m}$ that satisfies (2.2), (2.4), (2.6) and (2.10). We will need to use some properties of $R^{\dagger}_{n\times m}$ explained in section 2.

3. 1. Using Rectangular Matrix is of Full Column Rank

In this subsection we will use rectangular matrix $R_{m\times n}$ is of full column rank and will use the relations (2.1) and (2.2). So if we have the matrix of the plain text $P_{n\times r}$ then it is encrypted using the rectangular matrix $R_{m\times n}$ that is of full column rank as:

$$R_{m\times n}\, P_{n\times r} \mod M = C_{m\times r}\,, \tag{3.1}$$

and described using $R^{\dagger}_{n\times m}$ given by (2.1) as

$$R^{\dagger}_{n\times m}\ R_{m\times n}\, P_{n\times r} \mod M = R^{\dagger}_{n\times m} C_{m\times r} \mod M,$$

using (2.2) we obtain

$$I_{n\times n}\, P_{n\times r} \mod M = R^{\dagger}_{n\times m} C_{m\times r} \mod M,$$

this implies

$$P_{n\times r} = R^{\dagger}_{n\times m} C_{m\times r} \mod M, \tag{3.2}$$

thus, we obtain encryption (3.1) and description (3.2).

3. 2. Using Rectangular Matrix is of Full Row Rank

Here, we will use rectangular matrix $R_{m\times n}$ is of full column rank and will use the relations (2.3) and (2.4). So (by the same manner in the last subsection) if we have the matrix of the plain text $P_{r\times m}$ then it is encrypted using the rectangular matrix $R_{m\times n}$ that is of full row rank as:

$$P_{r\times m}\ R_{m\times n} \mod M = C_{r\times n}\,, \tag{3.3}$$

and described using $R^{\dagger}_{n\times m}$ given by (2.3) as

$$P_{r\times m}\ R_{m\times n}\ R^{\dagger}_{n\times m} \mod M = C_{r\times n}\, R^{\dagger}_{n\times m} \mod M,$$

using (2.4) we obtain

$$P_{r\times m}\ I_{m\times m} \mod M = C_{r\times n}\, R^{\dagger}_{n\times m} \mod M,$$

$$P_{r\times m} = C_{r\times n} R^{\dagger}_{n\times m} \mod M. \tag{3.4}$$

Thus, we obtain encryption

$$C_{r\times n} = P_{r\times m}\, R_{m\times n} \mod M,$$

and obtain description

$$P_{r\times m} = C_{r\times n} R^{\dagger}_{n\times m} \mod M.$$

3. 3. Using Rectangular Matrix is of Full Rank Factorization

In this subsection we will use a rectangular matrix

$$R_{m\times n} = E_{m\times k} F_{k\times n} \quad (R_{m\times n} = R, \quad E_{m\times k} = E, \quad F_{k\times n} = F),$$

where F is an orthogonal matrix ($F^T F = I$) with $rank(F) = k$. We will use $R = EF$ to convert the plain text $P(= P_{n\times r})$ into the cipher text $C(= C_{m\times r})$, and will use the relations $(2.5) - (2.10)$ to convert the cipher text C into the plain text P. But we must choose R to be in row echelon form, see Def.2.1.3. where F is an orthogonal matrix of $rank\ k$.

Now, if we have the matrix of the plain text P, we get encryption:

$$RP \mod M = C,$$

or

$$EFP \mod M = C.$$

For description, if we used R in encryption, we will use (2.5) for description. Let us use EF in encryption, so we use (2.6):

$$(EF)^{\dagger}EFP \mod M = (EF)^{\dagger}C \mod M$$
$$F^{\dagger}E^{\dagger}EFP \mod M = F^{\dagger}E^{\dagger}C \mod M$$
$$F^{\dagger}FP \mod M = F^{\dagger}E^{\dagger}C \mod M$$
$$FF^{\dagger}FP \mod M = FF^{\dagger}E^{\dagger}C \mod M$$
$$FP \mod M = FF^{\dagger}E^{\dagger}C \mod M$$
$$F^{T}FP \mod M = F^{T}FF^{\dagger}E^{\dagger}C \mod M$$
$$P = F^{\dagger}E^{\dagger}C \mod M.$$

Thus, we obtain encryption

$$C = RP = EFP \mod M$$

and description

$$P = F^{\dagger}E^{\dagger}C \mod M.$$

4. SECURITY ANALYSTS

Using the rectangular matrix that has the Moore-Penrose generalized inverse (2.1), (2.3), (2.5) *or* (2.7) *and* (2.9) in cryptography is more secure than using the square matrix which is invertible, because in the first using, the size of the matrix of the plain text and the cipher text is not equal. So the plain text is more difficult to describe because there is not one-to-one correspondence between the plain text and the cipher text. Also, we cannot analyze using the original encryption because R is a rectangular matrix, that is R is not invertible.

5. NUMERICAL EXAMPLES

In this section, we give some numerical examples to illustrate our results.

Example 5.1.

Asmaa wants to send a message to Zaleekha, it is

"CRYPTOGRAPHY IS COOL".

Note that, Asmaa must send the encryption key $R_{m\times n}$.

Encryption:

Asmaa will choose the encryption key (the matrix key):

$$R_{3\times 2} = \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 11 & 13 \end{pmatrix}$$

which is of full column rank, so it has a left inverse (the Moore-Penrose generalized inverse $R_{2\times 3}^{\dagger}$) satisfies (2.2), where

$$R_{2\times 3}^{\dagger} = \begin{pmatrix} \frac{22}{10} & \frac{24}{10} & \frac{14}{10} \\ 0 & \frac{20}{10} & \frac{24}{10} \end{pmatrix} = \begin{pmatrix} 10 & 5 & 4 \\ 0 & 2 & 5 \end{pmatrix}.$$

Asmaa uses Table 2, to convert the message or the plain text into numbers as follows:

Table 2. Conversion the plain text into numbers.

Letter	Number	Letter	Number	Letter	Number	Letter	Number	Letter	Number
C	2	Y	24	T	19	G	6	A	0
R	17	P	15	O	14	R	17	P	15

Letter	Number	Letter	Number	Letter	Number	Letter	Number
H	7	I	8	C	2	O	14
Y	24	S	18	O	14	L	11

So, she gets the matrix of the plain text $P_{2\times 9}$:

$$P_{2\times 9} = \begin{pmatrix} 2 & 24 & 19 & 6 & 0 & 7 & 8 & 2 & 14 \\ 17 & 15 & 14 & 17 & 15 & 24 & 18 & 14 & 11 \end{pmatrix}.$$

Now, Asmaa enciphers a plain text using (3.1):

$$R_{3\times 2}\, P_{2\times 9}\ (mod 26) = \begin{pmatrix} 1 & 17 & 9 & 5 & 19 & 1 & 10 & 18 & 21 \\ 25 & 17 & 11 & 19 & 1 & 21 & 10 & 4 & 17 \\ 9 & 17 & 1 & 1 & 13 & 25 & 10 & 22 & 11 \end{pmatrix} = C_{3\times 9}.$$

After that, Asmaa converts the matrix $C_{3\times9}$ into letter as follows (Table 3):

Table 3. Conversion the matrix of the plain text into letters.

Number	Letter	Number	Letter	Number	Letter	Number	Letter	Number	Letter
1	B	17	R	9	G	5	F	19	T
25	Z	17	R	11	L	19	T	1	B
9	G	17	R	1	B	13	N	13	N

Number	Letter	Number	Letter	Number	Letter	Number	Letter
1	B	10	K	18	S	21	V
21	V	10	K	4	E	17	R
25	Z	10	K	22	W	11	L

Asmaa sent

"BZGRRRGLBFTNTBNBVZKKKSEWVRL"

and the encryption key

$$R_{3\times2} = \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 11 & 13 \end{pmatrix}.$$

Description:

Zaleekha receives "BZGRRRGLBFTNTBNBVZKKKSEWVRL" and the encryption key

$$R_{3\times2} = \begin{pmatrix} 1 & 3 \\ 5 & 7 \\ 11 & 13 \end{pmatrix}.$$

Note that, Zaleekha will note that $R_{3\times2}$ is of full column rank, so she will use (2.2) and (3.2).

Zaleekha converts the letters of the message into numbers using Table 4, (she will use three rows because she will use $R_{2\times3}^{\dagger}\, C_{3\times r} = P_{2\times r}$).

Table 4. Conversion the cipher text into numbers.

Letter	Number	Letter	Number	Letter	Number	Letter	Number	Letter	Number
B	1	R	17	G	9	F	5	T	19
Z	25	R	17	L	11	T	19	B	1
G	9	R	17	B	1	N	13	N	13

Letter	Number	Letter	Number	Letter	Number	Letter	Number
B	1	K	10	S	18	V	21
V	21	K	10	E	4	R	17
Z	25	K	10	W	22	L	11

Zaleekha gets

$$C_{3\times 9} = \begin{pmatrix} 1 & 17 & 9 & 5 & 19 & 1 & 10 & 18 & 21 \\ 25 & 17 & 11 & 19 & 1 & 21 & 10 & 4 & 17 \\ 9 & 17 & 1 & 1 & 13 & 25 & 10 & 22 & 11 \end{pmatrix}.$$

After that, she uses (3.2):

$$P_{2\times 9} = R_{2\times 3}^{\dagger} C_{3\times 9} \quad (mod 26),$$
$$= \begin{pmatrix} 2 & 24 & 19 & 6 & 0 & 7 & 8 & 2 & 14 \\ 17 & 15 & 14 & 17 & 15 & 24 & 18 & 14 & 11 \end{pmatrix}.$$

Zaleekha converts the numbers of $P_{2\times 9}$ into letters (Table 5):

Table 5. Conversion the matrix of the cipher text into letters.

Number	Letter	Number	Letter	Number	Letter	Number	Letter	Number	Letter
2	C	24	Y	19	T	6	G	0	A
17	R	15	P	14	O	17	R	15	P

Number	Letter	Number	Letter	Number	Letter	Number	Letter
7	H	8	I	2	C	14	O
24	Y	18	S	14	O	11	L

Zaleekha gets "CRYPTUGRAPHY IS COOL".

Example 5.2.

Asmaa wants to send a message to Zaleekha, it is

"GO HOME".

Encryption:

Asmaa will choose the encryption key:

$$R_{2\times3} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \end{pmatrix}$$

$R_{2\times3}$ is of full row rank, so it has a right inverse (the Moore-Penrose generalized inverse $R^{\dagger}_{3\times2}$) satisfies (2.4), where:

$$R^{\dagger}_{3\times2} = \begin{pmatrix} \frac{6}{8} & \frac{-1}{8} \\ \frac{8}{8} & 0 \\ \frac{6}{8} & \frac{7}{8} \end{pmatrix} \quad (mode\ 26).$$

By the same away in last example, Asmaa uses Table 6.

Table 6. Conversion the plain text into numbers.

Letter	Number	Letter	Number
G	6	O	14
O	14	M	12
H	7	E	4

to get the matrix of the plain text $P_{3\times2}$, where

$$P_{3\times2} = \begin{pmatrix} 6 & 14 \\ 14 & 12 \\ 7 & 4 \end{pmatrix}.$$

Asmaa uses (3.3) to get

$$C_{3\times3} = \begin{pmatrix} 0 & 20 & 16 \\ 14 & 14 & 12 \\ 18 & 3 & 22 \end{pmatrix}.$$

After that, Asmaa converts the numbers of $C_{3\times3}$ into letters using Table 7, to get the enciphered message

Table 7. Conversion the matrix of the plain text into letters.

Number	Letter	Number	Letter	Number	Letter
0	A	20	U	16	Q
14	O	14	O	12	M
18	S	3	D	22	W

that is, Asmaa sent

"AOSUODQMW"

and the encryption key

$$R_{2\times3} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \end{pmatrix}$$

Description:

Zaleekha receives "AOSUODQMW" and the encryption key

$$R_{2\times3} = \begin{pmatrix} 2 & 3 & 4 \\ 1 & 2 & 5 \end{pmatrix}.$$

She converts the letters of the message into numbers using Table 8. Since Zaleekha will use (3.4) (because $R_{2\times3}$ is of full row rank), where $m = 2,\ n = 3$, then she will put the letters in three columns as follows:

Table 8. Conversion the cipher text into numbers.

Letter	Number	Letter	Number	Letter	Number
A	0	U	20	Q	16
O	14	O	14	M	12
S	18	D	3	W	22

Now, Zaleekha gets

$$C_{3\times3} = \begin{pmatrix} 0 & 20 & 16 \\ 14 & 14 & 12 \\ 18 & 3 & 22 \end{pmatrix}.$$

The relation (3.4) gives Zaleekha

$$P_{3\times2} = \begin{pmatrix} 6 & 14 \\ 14 & 12 \\ 7 & 4 \end{pmatrix}.$$

Now, Zaleekha converts the numbers of $P_{3\times2}$ into letters (Table 9):

Table 9. Conversion the matrix of the cipher text into letters.

Number	Letter	Number	Letter
6	G	14	O

14	O	12	M
7	H	4	E

Zaleekha gets "GO HOME".

6. CONCLUSIONS

Our study gave a new method in cryptography. It showed how to use the rectangular matrix and the Moore-Penrose generalized inverse for it in cryptography. In this method also, we showed how to choose the rectangular matrix as a key. We noted that using this method is more secure than using the square matrix which has inverse.

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