Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 145 (2020) 85-94 EISSN 2392-2192

Oblong Sum Labeling of Union of Some Graphs

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ABSTRACT

An oblong sum labeling of a graph $G = (V,E)$ with p vertices and q edges is a one to one function $f:V(G) \rightarrow \{0,2,4,6,...\}$ that induces a bijection $f^* : E(G) \rightarrow \{O_1, O_2, O_3,..., O_q\}$ of the edges of G defined by $f^*(uv) = f(u) + f(v)$ for all $e = uv \in E(G)$. The graph that admits oblong sum labeling is called oblong sum graph. In this article, the oblong sum labeling of union of some graphs are studied.

***Keywords*:** Oblong numbers, Oblong sum labeling, subdivision of graphs, union of graphs

AMS Classification: 05C78

1. INTRODUCTION

Graphs considered in this paper are finite, undirected and simple. Let G = (V,E) be a graph with p vertices and q edges. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (edges/both) then the labeling is called a vertex (edge/total) labeling. Rosa [15] introduced $\beta -$ valuation of a graph and Golomb [5] called it as graceful labeling. There are several types of graph labeling and a detailed survey is found in [10]. Harary [6] introduced the notion of sum graph and further various sum graphs were studied in [4, 7, 9, 19, 20]. Triangular sum labeling was introduced in [8] and further studied in [16, 17]. The concept of oblong sum labeling was introduced in [12] and further studied in [14]. Labeled graphs are becoming an increasing useful

(Received 11 April 2020; Accepted 29 April 2020; Date of Publication 30 April 2020)

family of mathematical models for a broad range of applications like designing X-Ray crystallography, formulating a communication network addressing system, determining an optimal circuit layouts, problems in additive number theory etc. A systematic presentation of diverse applications of graph labeling is given in [1-3, 11, 18]. Following definitions are necessary for the present study.

Definition 1.1: Let O_n be the nth oblong number. An oblong sum labeling of a graph $G = (V,E)$ with p vertices and q edges is a one to one function $f: V(G) \rightarrow \{0,2,4,6,...\}$ that induces a bijection $f^* : E(G) \rightarrow \{O_1, O_2, O_3,..., O_q\}$ of the edges of G defined *by* $f^*(uv) = f(u)+f(v)$ for all $e=uv \in E(G)$. The graph that admits oblong sum labeling is called oblong sum graph.

Definition 1.2: Let the graphs G_1 and G_2 have disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 respectively. Then their union $G = G_1UG_2$ is a graph with vertex set $V = V_1UV_2$ and edge set $E = E_1UE_2$. Clearly G_1UG_2 has p_1+p_2 vertices and q_1+q_2 edges.

Definition 1.3: A subdivision of an edge $e = uv$ of a graph G is the replacement of the edge e by a path (u,w,v). If every edge of G is subdivided exactly once, then the resulting graph is called the subdivision graph $S(G)$.

Definition 1.4 [13]: The bistar $B_{m,n}$ is a graph obtained from K_2 by joining m pendant edges to one end of K_2 and n pendant edges to the other end of K_2.

2. MAIN RESULTS

Observation 2.1: There does not exist consecutive integers which are oblong numbers.

Observation 2.2: There does not exist consecutive oblong numbers whose difference is two.

Proof: Difference of consecutive oblong numbers is (n+1)(n+2) - n(n+1) = 2(n+1) > 2.

Lemma 2.3: In every oblong sum graph G, the vertices with label 0 and 2 are always adjacent.

Proof: The edge label $O_1 = 2$ is possible only when the vertices with label 0 and 2 are adjacent.

Lemma 2.4: In any oblong sum graph G, 0 and 2 cannot be the label of vertices of the same triangle contained in it.

Proof: Let a_0, a_1 and a_2 be the vertices of a cycle a_0 and a_1 are labeled with 0 and 2 respectively and a_2 is labeled with some x $\in N$, where $x \neq 0, x \neq 2$. Such vertex labeling will give rise to edge labels 2, x, and x+2. In order to admit an oblong sum labeling, x and x+2 must be oblong sum numbers. But it is not possible by observation 2.2.

Lemma 2.5: In any oblong sum graph G, 2 and 4 cannot be the labels of the vertices of the same cycle in G.

Proof: Let a_0, a_1 and a_2 be the vertices of a cycle a_0 and a_1 are labeled with 2 and 4 respectively and a_2 is labeled with some x ∈ N, where $x \neq 2, x \neq 4$. Such vertex labeling will give rise to edge labels 6, x+2 and x+4. In order to admit a oblong sum labeling, $x + 2$ and $x + 4$ must be oblong sum numbers which is not possible by observation 2.2.

Theorem 2.6: $S(K_{1,n})$ is oblong sum for all n ≥ 1.

Proof: Let u,u_i, v_i, $1 \leq i \leq n$ be the vertices of $S(K_{1,n})$ and uu_i,$u_i v_i$, $1 \leq i \leq n$ be the edges of $S(K_{1,n})$.

Let $f: S(K_{1,n}) \rightarrow \{0,2,4,6, \dots.\}$ be defined as follows.

$f(u) = 0$

$f(u_i) = O_i$,$1 \leq i \leq n$

$f(v_i) = O_{n+i} - f(v_i), 1 \leq i \leq n$

Let f^* be the induced edge labeling of f. Then

$f^*(uu_i) = O_i$,$1 \leq i \leq n$

$f^*(u_i v_i) = O_{n+i}$,$1 \leq i \leq n$

The induced edge labels are distinct and are $O_1, O_2, O_3, \dots., O_{2n}$.
Hence $S(K_{1,n})$ is oblong sum.

Example 2.7: oblong sum labeling of $S(K_{1,4})$ is given in Fig. 1.

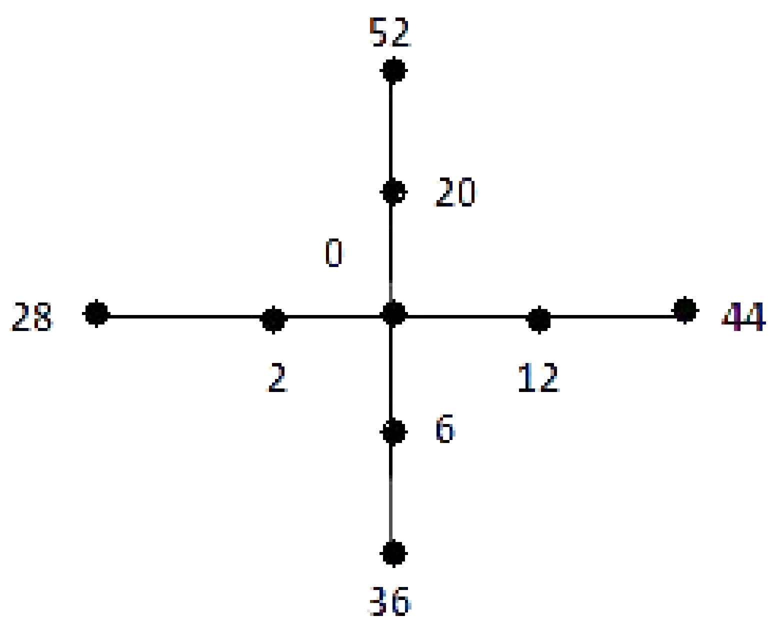

Fig. 1.

Theorem 2.8: $S(K_{1,n}) \cup K_{1,m}$ is oblong sum for all $n, m > 1$.

Proof: Let $u, u_i, v_i, w, w_j, 1 \le i \le n, 1 \le j \le m$ be the vertices of $S(K_{1,n}) \cup K_{1,m}$ and $uu_i, u_iv_i, ww_j, 1 \le i \le n, 1 \le j \le m$ be the edges of $S(K_{1,n}) \cup K_{1,m}$.

Let $f: (S(K_{1,n}) \cup K_{1,m}) \to \{0,2,4,6, \dots.\}$ be defined as follows.

$f(u) = 0$

$f(u_i) = O_i, 1 \le i \le n$

$f(v_i) = O_{n+i} - O_i, 1 \le i \le n$

$f(w) = f(v_{n-2}) - 2$

$f(w_j) = O_{2n+j} - f(w), 1 \le j \le m$

Let f^* be the induced edge labeling of *f*. Then

$f^*(uu_i) = O_i, 1 \le i \le n$

$f^*(u_iv_i) = O_{n+i}, 1 \le i \le n$

$f^*(ww_j) = O_{2n+j}, 1 \le j \le m$

The induced edge labels are distinct and are $O_1, O_2, O_3, \dots., O_{2n+m}$.
Hence $S(K_{1,n}) \cup K_{1,m}$ is oblong sum.

Example 2.9: Oblong sum labeling of $S(K_{1,4}) \cup K_{1,5}$ given in Fig. 2.

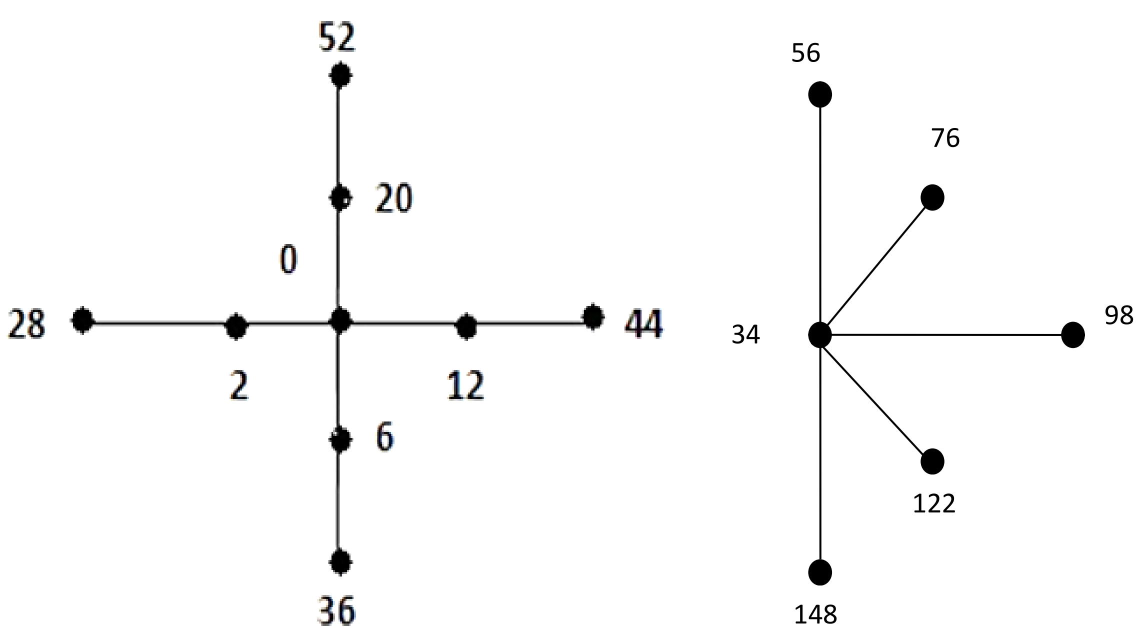

Fig. 2.

Theorem 2.10: $S(K_{1,n}) \cup B_{r,s}$ is oblong sum for all $n, r, s > 1$.

Proof: Let u,u_i, v_i,w, w_j, x, x_k $1 \leq i \leq n$, $1 \leq j \leq r$, $1 \leq k \leq s$ be the vertices of $S(K_{1,n}) \cup B_{r,s}$ and uu_i,u_iv_i,ww_j, wx, xx_k, $1 \leq i \leq n$, $1 \leq j \leq r$, $1 \leq k \leq s$ be the edges of $S(K_{1,n}) \cup B_{r,s}$.

Let $f: S(K_{1,n}) \cup B_{r,s} \rightarrow \{0,2,4,6, \dots\}$ be defined as follows.

$f(u) = 0$

$f(u_i) = O_i$,$1 \leq i \leq n$

$f(v_i) = O_{n+i} - O_i$, $1 \leq i \leq n$

$f(w) = f(v_{n-2}) - 2$

$f(w_j) = O_{2n+j+1} - f(w)$, $1 \leq j \leq r$

$f(x) = O_{2n+1} - f(w)$

$f(x_k) = O_{2n+m+1+k} - f(x)$, $1 \leq k \leq s$

Let f^* be the induced edge labeling of f. Then

$f^*(uu_i) = O_i$,$1 \leq i \leq n$

$f^*(u_iv_i) = O_{n+i}$,$1 \leq i \leq n$

$f^*(ww_j) = O_{2n+j+1}$, $1 \leq j \leq r$

$f^*(wx) = O_{2n+1}$

$f^*(xx_k) = O_{2n+r+1+k}$,$1 \leq j \leq r$, $1 \leq k \leq s$

The induced edge labels are distinct and are $O_1, O_2, O_3, \dots, O_{2n+r+s+1}$.
Hence $S(K_{1,n}) \cup B(r,s)$ is an oblong sum graph.

Example 2.11: Oblong sum labeling of $K_{1,3} \cup B_{3,5}$ is given in Fig. 3.

Theorem 2.12: $S(K_{1,n}) \cup S(K_{1,m})$ is oblong sum for all $n, m > 1$.

Proof: Let u,u_i, v_i,w, w_j, x_j $1 \leq i \leq n$, $1 \leq j \leq m$ be the vertices of $S(K_{1,n}) \cup S(K_{1,m})$ and uu_i,u_iv_i,ww_j, w_jx_j, $1 \leq i \leq n$, $1 \leq j \leq m$ be the edges of $S(K_{1,n}) \cup S(K_{1,m})$.

Let $f: S(K_{1,n}) \cup S(K_{1,m}) \rightarrow \{0,2,4,6, \dots\}$ be defined as follows.

$f(u) = 0$

$f(u_i) = O_i$,$1 \leq i \leq n$

$f(v_i) = O_{n+i} - f(u_i)$, $1 \leq i \leq n$

$f(w) = f(v_{n-2}) - 2$

$f(w_j) = O_{2n+j} - f(w)$, $1 \leq j \leq m$

$f(x_j) = O_{2n+m+j} - f(w_j)$, $1 \leq j \leq m$

Let f^* be the induced edge labeling of f. Then

$f^*(uu_i) = O_i$,$1 \leq i \leq n$

$f^*(u_i v_i) = O_{n+i}$,$1 \leq i \leq n$

$f^*(ww_j) = O_{2n+j+1}, 1 \leq j \leq m$

$f^*(w_j x_j) = O_{2n+m+j}$,$1 \leq j \leq m$

The induced edge labels are distinct and are $O_1, O_2, O_3, \dots, O_{2n+2m}$.
Hence $S(K_{1,n}) \cup S(K_{1,m})$ is an oblong sum graph.

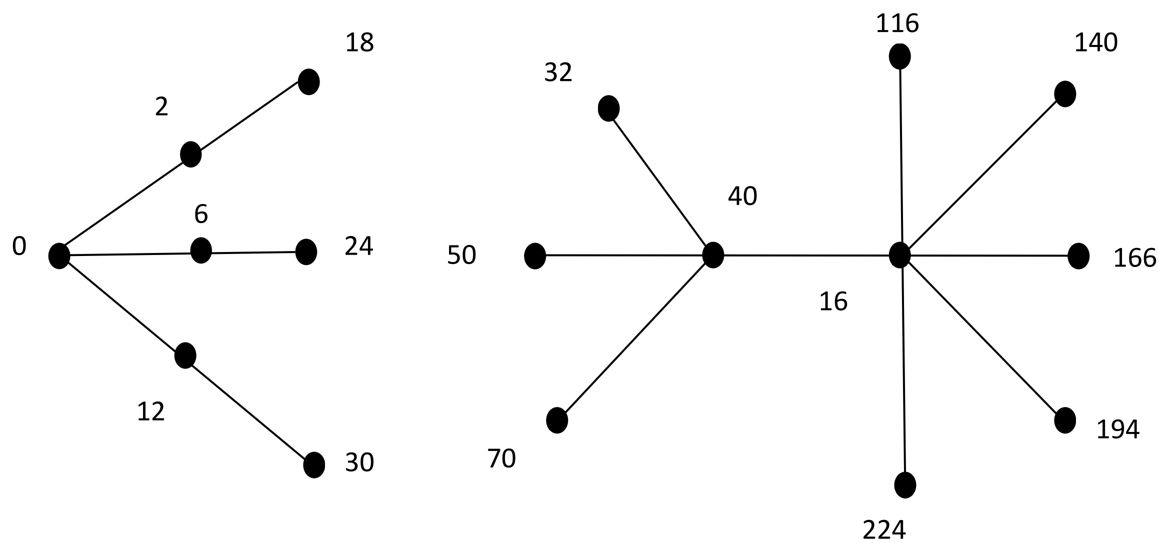

Fig. 3.

Example 2.13: Oblong sum labeling of $S(K_{1,4}) \cup S(K_{1,3})$ is given in Fig. 4.

Theorem 2.14: $K_{1,n} \cup B(m,r)$ is oblong sum for all $n > 2$,m, r ≥ 1

Proof: Let $u, u_i, v, v_j, w, w_s, 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq s \leq r$ be the vertices and $uu_i, vv_j, vw, ww_k, 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq r$be the edges of $K_{1,n} \cup B(m,r)$.

Let $f: V(K_{1,n} \cup B(m,n)) \to \{0,2,4,6, \dots\}$ be defined as follows

$f(u) = 0$

$f(u_i) = O_i$,$1 \leq i \leq n$

$f(v) = O_{n+1} - 4$

$f(v_j) = O_{n+j} - f(v), 1 \leq j \leq m$

$f(w) = O_{n+j+1} - f(v), 1 \leq j \leq m$

$f(w_k) = O_{n+m+1+k} - f(w), 1 \leq k \leq r$

Let f^* be the induced edge labeling of f. Then

$f^*(uu_i) = O_i; 1 \leq i \leq n$

$f^*(vv_j) = O_{n+j}, 1 \leq j \leq m$

$f^*(vw) = O_{n+j+1}, 1 \leq j \leq m$

$f^*(ww_k) = O_{n+m+1+k}, 1 \leq k \leq r$

The induced edge labels are distinct and are $O_1, O_2, O_3, \ldots, O_{n+m+r+1}$.
Hence $K_{1,n} \cup B(m,n)$ is an oblong sum graph.

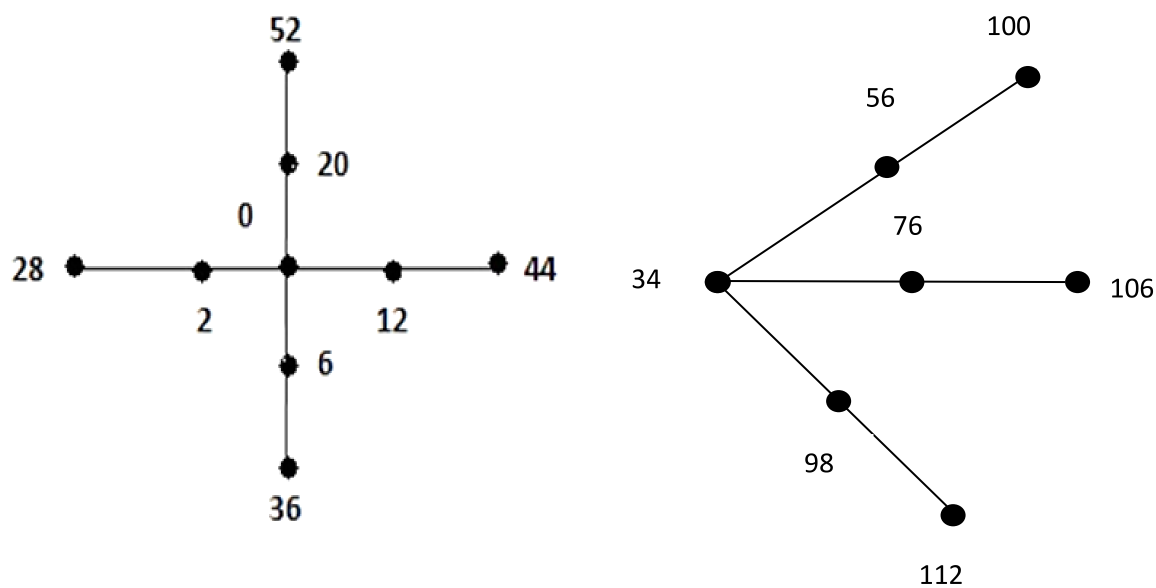

Fig. 4.

Example 2.15: Oblong sum labeling of $K_{1,5} \cup B(3,4)$ is given in Fig. 5.

Theorem 2.16: The helm graph H_n is not a oblong sum graph.

Proof: Let us denote the apex vertex asC_1, the consecutive vertices adjacent to C_1 as $v_1, v_2, \ldots, v_n$. Let the pendant vertices adjacent to $v_1, v_2, \ldots, v_n$ be $u_1, u_2, \ldots, u_n$ respectively. Suppose, H_n admits a oblong sum labeling. Suppose $f : V(H_n) \to \{0,2,4,6, \ldots\}$ be an oblong sum labeling. Now there exists two cases.

Case 1: Suppose $f(C_1) = 0$.
Then according to lemma 2.3, we have to assign label 2 exactly one of the vertices from $v_1, v_2 \ldots, v_n$. Then there is a triangle having the vertices with labels 0 and 2 as adjacent vertices, which contradicts the lemma 2.4.

Case 2: Any one of the vertices from $v_1, v_2 \ldots, v_n$is labeled with 0. Without loss of generality let us assume that $f(v_1) = 0$. Since each of the vertices from c_1, v_2, v_n, u_1 is adjacent to v_1, one of the vertices from them must be labeled with 2.

Subcase 1: Suppose one of the vertices from c_1, v_2, v_n is labeled with 2. In each possibility there is a triangle having two of the vertices with labels 0 and 2, which contradicts the lemma 2.4.

Subcase 2: Suppose $f(u_1) = 2$. Now, the edge label $O_2 = 6$ can be obtained by vertex labels 0, 6 or 2, 4. The vertex with label 2 and the vertex label 4 cannot be adjacent as u_1 is a pendant vertex having label 2 and it is adjacent to the vertex with label 0.Therefore one of the vertices from v_2, v_n, c_1 must recive the label 6. Thus there is a triangle whose two of the vertices are labeled with 0 and 6. Let the third vertex be labeled with x, with $x \neq 0$ and $x \neq 6$. To admit a oblong sum labeling $x, x + 6$ are two distinct oblong numbers other than 6 having difference 6, which is not possible.

Case 3: Any one of the vertices from $u_1, u_2, \ldots, u_n$ is labeled with 0. Without loss of generality, we may assume that $f(u_1) = 0$. Then according to lemma 2.3, $f(v_1) = 2$. The edge label $O_2 = 6$ can be obtained by vertex labels 0,6 or 2,4. The vertex with label 0 and the vertex with label 6 cannot be the adjacent vertices as u_1 is a pendant vertex having label 0. and it is adjacent to the vertex with label 2. Therefore one of the vertices from v_2, v_n, c_1 must be labelled with 6. Thus we have a triangle having vertices with labels 2 and 4 which contradicts the lemma 2.5. Thus in each of the above cases discussed above, H_n is not oblong sum.

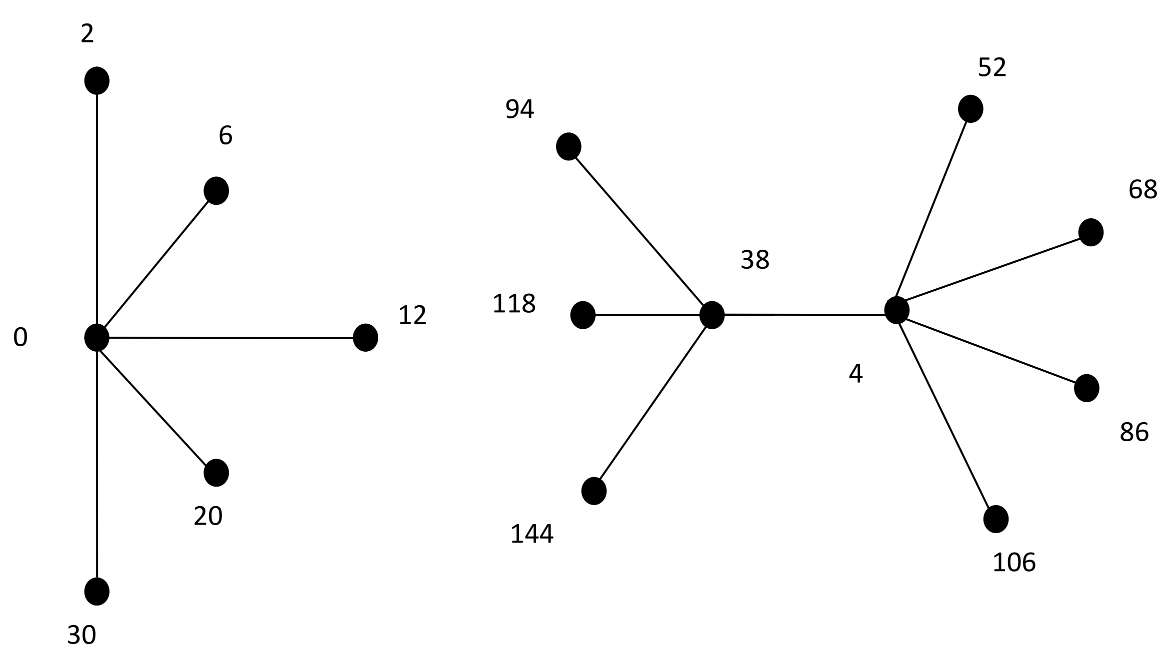

Fig. 5.

3. CONCLUSION

In this paper, the authors studied the oblong sum labeling of union of some graphs, subdivision of a star and also proved that helm is not oblong sum. Further studies can be made on various graphs.

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