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Nonlinear Thermally Induced Dynamic Analysis of Non-Homogenous Rectangular Plate with Varying Thickness Using Three-Dimensional Differential Transform Method

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ABSTRACT

The practical importance of thermally induced varying thickness plates has recently become an area of increase interest for engineers due to its wide applications. The temperature effect alters the modulus of elasticity of the plate causing an irrational behavior of the plate. This present study presents the application of three-dimensional differential transform method (3D-DTM) to nonlinear thermally induced dynamic analysis of non-homogenous rectangular plate with varying thickness under external excitation. Three-dimensional differential transform is used to obtain the analytical solution to the governing differential equation and the solution is used for the parametric studies. It is shown that, taper constant increases with increase in maximum deflection, thermal constant increases with decreases in maximum deflection, increases in aspect ratio leads to decreases in maximum deflection, increase in natural frequency results to increases in maximum deflection and non-homogeneity constant increases with increase in maximum deflection. Findings of the research is expected to add value to existing knowledge of classical plate theory.

Keywords**:** Three-dimensional differential transform, thermal induced dynamic vibration, Non-Homogenous, varying thickness

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1. INTRODUCTION

Nonlinear thermal induced dynamic analysis of non-homogenous plate with linearly varying thickness under external force has recently gained research attention and investigations. Variable thickness plate under thermal effect and excitation force has economic advantage; the thermal effect is significant in certain industrial uses, such as, space technology application, combustion process, internal combustion engine, ship compressors. Research into the topic has gained special attention due to real-life application of the topic in industries. During machine operation, plates are subjected to heat fluxes thereby affecting the modulus of elasticity, study of thermal effect on modulus of elasticity is very important. In an attempt to determine the natural frequency of varying thickness plate in 1977 Jain and Soni [1] worked on parabollic plate, following year, Irie and Yamada [2] analyzed circular plate under thermal induced.

Subsequent year, Laura et al. [3] studied transverse vibration of rectangular plate under varying thickness. Four years later, Tomar and Gupta [4, 5] determined the natural frequency of linearly varying thickness plate under thermal effect. Chang et al. [6] adopted numerical method in the study of thermally induced vibration of plate. Singh and Saxena [7] conducted research on rectangular plate with bidirection thickness varaiation.

Kawamura et al. [8] conducted research using exact method on thermo-elasticity equations for thermally induced plate. Gupta et al. [9, 10] examined using exact method thermal effect on vibration of non-homogenous rectangular with bi-directional parabolic varying thickness. Khanna and Singhal [11] Considered analytical approach on thermally induced dynamic behaviour of nonhomogenous tapered rectangular plate. Based on the disadvantages of convergenece study, huge computational time and cost associated with numerical method examined by [12-14]. Also, the exact soution investiagted by [8-10] accompanying with limitation of linear governing equation and sound mathmatics principle application. Thereby prompt the adoption of semi-analytical method which has taken care of both limitations in numerical and exact method. Gupta et al. [15] investiagated dynamic analysis of nonhomogenous circular plate using Quadrature method, Rashidi et al. [16] used homotopy perturbation method (HPM) for nonlinear vibration of rectangular plate. Likewise, Marinca and herisanu [17] adopted Optimal homotopy asymptotic method (OHAM). Though, HPM and OHAM are very effective semi-analytical method but still have limitation of finding auxiliary parameter which also affect the computational time and cost. The use of differential transform method [18] provides easier and simpler approach to solving the problems.

Reviewed literature shows that application of three-dimensional differential transform method to nonlinear thermally Induced dynamic analysis of non-homogenous rectangular plate with varying thickness in presence of external excitation to authors knowledge has not been conducted. Hence, the research focuses on application of three-dimensional differential transform method to nonlinear thermally induced dynamic analysis of non-homogenous rectangular plate with varying thickness in presence of external excitation

2. PROBLEM FORMULATION AND MATHEMATICAL ANALYSIS

Considering a rectangular plate of symmetric form. Let a represent the length and b represent the width of the plate. The boundary condition is simply supported along the two opposite site $y=0$ and $y=b$, and free alongside $x=0$ and $x=a$. Plate is assumed subjected to one dimensional temperature distribution along the length

$$T = T_o\left(1-\frac{1}{2}\left(\frac{x}{a}+\frac{x^2}{a^2}\right)\right); \tag{1}$$

where T represents the temperature excess above reference temperature at any point $\frac{x}{a}$ and T_o represents temperature excess above reference temperature at $x=0$.

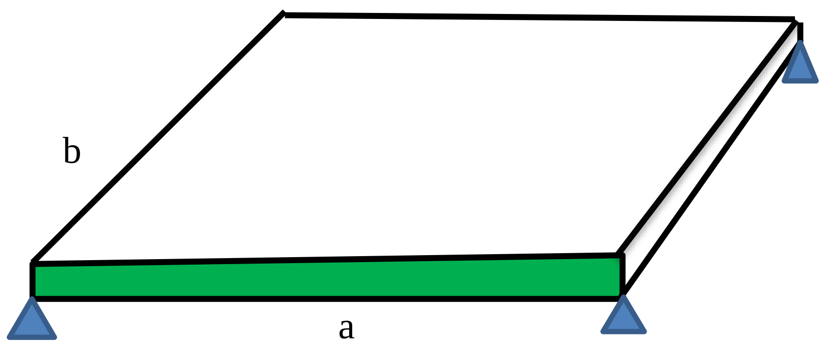

Fig. 1. Showing rectangular plate of variable thickness

Since temperature, distribution affects modulus of elasticity *(E)*

$$E = E_o\left(1-\gamma T\right); \tag{2}$$

where E_o is the Young's modulus at reference temperature, γ is the slope variation of *E* and *T*. Modulus variation becomes;

$$E = E_o\left(1-\alpha\left(1-\frac{1}{2}\left(\frac{x}{a}+\frac{x^2}{a^2}\right)\right)\right); \tag{3}$$

where $\alpha=\gamma T_o\ (0\le\alpha\le 1)$ thermal constant parameter. The thickness h variation is assumed linear along $x-direction$ as

$$h = h_o\left(1+\beta\frac{x}{a}\right); \tag{4}$$

where h_o is the plate thickness at $x=0$, and β is taper constant. The density ρ of the plate is assumed to vary linearly in n x-direction also as

$$\rho = \rho_o\left(1+\alpha_1\frac{x}{a}\right); \tag{5}$$

where ρ_o is the initial density and α is non-homogenous parameter.

The governing equation of a thin rectangular plate in Cartesian coordinate is;

$$\frac{\partial^2}{\partial x^2}\left[-D\left(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}\right)\right]+2\frac{\partial^2}{\partial x\partial y}\left[-D(1-v)\frac{\partial^2 w}{\partial x\partial y}\right]+\frac{\partial^2}{\partial y^2}\left[-D\left(\frac{\partial^2 w}{\partial y^2}+v\frac{\partial^2 w}{\partial x^2}\right)\right]=h\rho\frac{\partial w}{\partial t}+ Q_o(x,y)\sin\omega t; \tag{6}$$

$$\mu h^3\left(\frac{\partial^4 w}{\partial x^4}+2v\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)+2\mu\frac{\partial h^3}{\partial x}\left(\frac{\partial^3 w}{\partial x^3}+v\frac{\partial^3 w}{\partial y^2\partial x}\right)+\mu\frac{\partial^2 h^3}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}\right)=\rho h\frac{\partial w}{\partial t}+ Q_o(x,y)\sin\omega t; \tag{7}$$

where $D=\dfrac{Eh^3}{12\left(1-v^2\right)}$ is a flexural rigidity, w is deflection, h is the thickness of the plate, t is the time, E is the modulus of elasticity, v is the Poisson's ratio (Constant), ρ is the mass density per unit volume, $Q_o(x,y)\sin\omega t$ is the excitation force.

Nondimensional term $\xi=\dfrac{x}{a}$ and $\eta=\dfrac{y}{b}$

Substitute $x=\xi a$ and $y=\eta b$ into Eq. (7)

$$\mu h^3\left(b^4\frac{\partial^4 W}{\partial\xi^4}+2va^2b^2\frac{\partial^4 W}{\partial\eta^2\partial\xi^2}+a^4\frac{\partial^4 W}{\partial\eta^4}\right)+2\mu\frac{\partial h^3}{\partial\xi}\left(b^4\frac{\partial^3 W}{\partial\xi^3}+va^2b^2\frac{\partial^3 W}{\partial\eta^2\partial\xi}\right)+ \mu\frac{\partial^2 h^3}{\partial\xi^2}\left(b^4\frac{\partial^2 W}{\partial\xi^2}+va^2b^2\frac{\partial^2 W}{\partial\eta^2}\right)=a^4b^4\rho h\frac{\partial W}{\partial t}+Q_o(x,y)\sin\omega t; \tag{8}$$

$$\left(1-\alpha\left(1-\frac{1}{2}\left(\frac{x}{a}+\frac{x^2}{a^2}\right)\right)\right)\mu\left(h_o\left(1+\beta\frac{\xi}{a}\right)\right)^3\left[b^4\frac{\partial^4 W}{\partial\xi^4}+2a^2b^2\frac{\partial^4 W}{\partial\xi^2\partial\eta^2}+a^4\frac{\partial^4 W}{\partial\eta^4}\right]+ 2\mu\frac{\partial\left(\left(1-\alpha\left(1-\frac{1}{2}\left(\frac{x}{a}+\frac{x^2}{a^2}\right)\right)\right)h_o\left(1+\beta\frac{\xi}{a}\right)\right)^3}{\partial\xi}\left[b^4\frac{\partial^3 W}{\partial\xi^3}+a^2b^4\frac{\partial^3 W}{\partial\xi\partial\eta^2}\right]+ \frac{\partial\left(\left(1-\alpha\left(1-\frac{1}{2}\left(\frac{x}{a}+\frac{x^2}{a^2}\right)\right)\right)\left(h_o\left(1+\beta\frac{\xi}{a}\right)\right)^3\right)^2}{\partial\xi^2}\left[b^4\frac{\partial^2 W}{\partial\xi^2}+a^2b^4v\frac{\partial^2 W}{\partial\eta^2}\right]- a^4b^4\rho_o h_o\left(1+\alpha_1\frac{\xi}{a}\right)\left(1+\beta\frac{\xi}{a}\right)\omega^2[W]=Q_o(x,y)\sin\omega t; \tag{9}$$

Assuimg $\mu = \dfrac{E}{12\left(1-v^2\right)}$; Therefore, flexural ridigity $D = \mu h^3$

2. 1. Boundary Condition

- Simply Supported edge [20]

For $x=0$, and $x=a$ are;

$$w=0, \quad \frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2} = 0; \tag{10}$$

And for $y=0$, and $y=\mathrm{b}$ are;

$$w=0, \quad \frac{\partial^2 w}{\partial y^2} + v\frac{\partial^2 w}{\partial x^2} = 0; \tag{11}$$

- Free edge

For $x=0$, and $x=a$ are;

$$\frac{\partial^2 w}{\partial x^2} + v\frac{\partial^2 w}{\partial y^2} = 0,$$

(12)

And for $y=0$, and $y=\mathrm{b}$ are;

$$\frac{\partial^3 w}{\partial x^3} + \left(2-v\right)\frac{\partial^3 w}{\partial x \partial y^2} = 0; \tag{13}$$

The plate is assumed subjected to one-dimensional steady temperature distribution along the length:

Table 1. Showing Material properties.

Young's Modulus	Poison's ratio	mass density	Initial thickness	length
E	ν	ρ_m [kg/m^3]	h_o [mm]	a [m]
200,000	0.3	8,000	5	1

3. METHOD OF SOLUTION: THREE DIMENSION DIFFERENTIAL TRANSFORM METHOD

One dimensional Differential Transform method introduced by Zhou [21], was used to solve initial ordinary differential electric circuit equation. Chen and Ho [22] later applied it to Partial differential equation and confirmed predicting very reliable solution. The procedure for the method is stated as [23];

$$w(x,y,t)=W(k,h,m)=\frac{1}{k!h!}\left[\frac{\partial^{k+h+m}w(x,y,t)}{\partial x^k \partial y^h \partial t^m}\right]_{[0,0,0]_m} ; \tag{14}$$

Differential inverse transform

$$w(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{p=0}^{\infty}W(k,h)x^k y^h t^m,; \tag{15}$$

From Eq. (13) and (14) leads to

$$w(x,y,t)=\sum_{k=0}^{\infty}\sum_{h=0}^{\infty}\sum_{p=0}^{\infty}\frac{1}{k!h!m!}\left[\frac{\partial^{k+h+m}w(x,y,t)}{\partial x^k \partial y^h \partial t^m}\right]_{[0,0,0]} x^k y^h t^m, \tag{16}$$

Table 2. Showing Differential transformation method Operational properties.

Original function	Transformed function
$x^o y^n t^s$	$\delta(k-o,h-n,m-s)=\begin{vmatrix}1 & k=0,h=n,m=s\\ 0 & otherwise\end{vmatrix}$
$\frac{\partial w(x,y,t)}{\partial x}$	$(k+1)W(k+1,h,m)$
$\frac{\partial w(x,y)}{\partial t}$	$(h+1)W(k,h,m+1)$
$\frac{\partial w(x,y,t)}{\partial x}\frac{\partial w(x,y,t)}{\partial y}$	$\sum_{l=0}^{k}\sum_{s=0}^{h}\sum_{p=0}^{m}(k-r+1)\,(h-s+1)\,W(k-r+1,s,p)$
$x^m \cos(\propto t+\beta)$	$\frac{\alpha^m}{m!}\delta(k,m)\cos\left(\frac{m\pi}{2}+\beta\right)$
$\frac{\partial^{r+s+p}w(x,y,t)}{\partial x^r \partial y^s \partial t^p}$	$(k+1)(k+2)\ldots..(k+r)(h+1)(h+2)\ldots..(h+s)(m+1)(m+2)\ldots\ldots(m+p)w(k+r,h+s,m+p)$

$$\text{Natural frequency } = \frac{\omega a^2}{\pi^2}\sqrt{\frac{\rho h}{D}} \tag{17}$$

where ρ is the density, h is the thickness, D is the flexural rigidity, ω is frequency

Table 2 contains some differential transform of some functions.

3. 1. Application of three-dimension differential transform method to the governing equation under investigation

Having expanded and simply Eq. (9), we have

$$\begin{aligned}
&\left\{\begin{array}{l}
\mu h^3 + \frac{3\mu h^3\beta\xi}{a} + \frac{3\mu h^3\beta^2\xi^2}{a^2} + \frac{\mu h^3\beta^3\xi^3}{a^3} - \mu h^3\alpha - \frac{3\mu h^3\alpha\beta\xi}{a} \\
-\frac{3\mu h^3\alpha\beta^2\xi^2}{a^2} - \frac{\mu h^3\alpha\beta^3\xi^3}{a^3} + \frac{1}{2}\frac{\mu h^3\alpha\xi}{a} + \frac{3}{2}\frac{\mu h^3\alpha\xi^2\beta}{a^2} + \frac{3}{2}\frac{\mu h^3\alpha\xi^3\beta^2}{a^3} \\
+\frac{1}{2}\frac{\mu h^3\alpha\xi^4\beta^3}{a^4} + \frac{1}{2}\frac{\mu h^3\alpha\xi^2}{a^2} + \frac{3}{2}\frac{\mu h^3\alpha\xi^3\beta}{a^3} + \frac{3}{2}\frac{\mu h^3\alpha\xi^4\beta^2}{a^4} + \frac{1}{2}\frac{\mu h^3\alpha\xi^5\beta^3}{a^5}
\end{array}\right\}\left(b^4\frac{\partial^4 W}{\partial\xi^4} + 2va^2b^2\frac{\partial^4 W}{\partial\eta^2\partial\xi^2} + a^4\frac{\partial^4 W}{\partial\eta^4}\right) \\
&\left\{\begin{array}{l}
\frac{\mu h^3\alpha}{a} + \frac{6\mu h^3\alpha\beta\xi}{a^2} + \frac{9\mu h^3\alpha\beta^2\xi^2}{a^3} + \frac{4\mu h^3\alpha\beta^3\xi^3}{a^4} + \frac{2\mu h^3\alpha\xi}{a^2} + \frac{9\mu h^3\alpha\xi^2\beta}{a^3} \\
+\frac{12\mu h^3\alpha\xi^3\beta^2}{a^4} + \frac{5\mu h^3\alpha\xi^4\beta^3}{a^5} + \frac{6\mu h^3\beta}{a} + \frac{12\mu h^3\beta^2\xi}{a^2} + \frac{6\mu h^3\beta^3\xi^2}{a^3} - \frac{6\mu h^3\alpha\beta}{a} \\
-\frac{12\mu h^3\alpha\beta^2\xi}{a^2} - \frac{6\mu h^3\alpha\beta^3\xi^2}{a^3}
\end{array}\right\}\left(b^4\frac{\partial^3 W}{\partial\xi^3} + va^2b^2\frac{\partial^3 W}{\partial\eta^2\partial\xi}\right) \\
&\left\{\begin{array}{l}
\frac{\mu h^3\alpha}{a^2} + \frac{9\mu h^3\alpha\beta\xi}{a^3} + \frac{18\mu h^3\alpha\beta^2\xi^2}{a^4} + \frac{10\mu h^3\alpha\beta^3\xi^3}{a^5} + \frac{3\mu h^3\alpha\beta}{a^2} + \frac{9\mu h^3\alpha\beta^2\xi}{a^3} \\
+\frac{6\mu h^3\alpha\beta^3\xi^2}{a^4} + \frac{6\mu h^3\beta^2}{a^2} + \frac{6\mu h^3\beta^3\xi}{a^3} - \frac{6\mu h^3\alpha\beta^2}{a^2} - \frac{6\mu h^3\alpha\beta^3\xi}{a^3}
\end{array}\right\}\left(b^4\frac{\partial^2 W}{\partial\xi^2} + va^2b^2\frac{\partial^2 W}{\partial\eta^2}\right) \\
&-a^4b^4\rho_o h_o\left(1+\sigma\frac{x}{a}\right)\left(1+\beta\frac{x}{a}\right)\frac{\partial^2 W}{\partial\tau^2} = Q_o(x,y)\sin\omega t
\end{aligned} \tag{18}$$

Applying the principle of DTM stated in Table 1 to Eq. (18)

$$\mu h_o^3 a^4 (h+1)(h+2)(h+3)(h+4)W[k,h+4,m]$$
$$+3a^3\mu h_o^3\beta\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+3a^2\mu h_o^3\beta^2\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+a\mu h_o^3\beta^3\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$-\mu h_o^3\alpha a^4 (h+1)(h+2)(h+3)(h+4)W[k,h+4,m]$$
$$-3a^3\mu h_o^3\alpha\beta\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$-3a^2\mu h_o^3\alpha\beta^2\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$-a\mu h_o^3\alpha\beta^3\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+\frac{1}{2}a^3\mu h_o^3\alpha\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+\frac{3}{2}a^2\mu h_o^3\alpha\beta\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+\frac{3}{2}a\mu h_o^3\alpha\beta^2\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$

$$+\frac{1}{2}\mu h_o^3\alpha\beta^3\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-4,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+\frac{1}{2}a^2\mu h_o^3\alpha\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+\frac{3}{2}a\mu h_o^3\alpha\beta\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+\frac{3}{2}\mu h_o^3\alpha\beta^2\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-4,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+\frac{1}{2}\frac{\mu h_o^3\alpha\beta^3}{a}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-5,h-p,m-s)(p+1)(p+2)(p+3)(p+4)W[k-l,p+4,s]$$
$$+\frac{4vb^2\mu h_o^3\alpha\beta^3}{a^2}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$+2vb^2\mu h_o^3\alpha\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$+\frac{9vb^2\mu h_o^3\alpha\beta}{a}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$+\frac{b^4\mu h^3\alpha}{a}(k+1)(k+2)(k+3)W[k+3,h,m]$$
$$+\frac{6b^4\mu h^3\alpha\beta}{a^2}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$+\frac{9b^4\mu h^3\alpha\beta^2}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$+\frac{4b^4\mu h^3\alpha\beta^3}{a^4}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$+\frac{2b^4\mu h^3\alpha}{a^2}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$+\frac{9b^4\mu h^3\alpha\beta}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$+\frac{12b^4\mu h^3\alpha\beta^2}{a^4}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$+\frac{5b^4\mu h^3\alpha\beta^3}{a^5}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-4,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$+\frac{6b^4\mu h^3\beta}{a}(k+1)(k+2)(k+3)W[k+3,h,m]$$
$$+\frac{12b^4\mu h^3\beta^2}{a^2}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$+\frac{6b^4\mu h^3\beta^3}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$-\frac{6b^4\mu h^3\alpha\beta}{a}(k+1)(k+2)(k+3)W[k+3,h,m]$$

$$-\frac{12b^4\mu h^3\alpha\beta^2}{a^2}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$-\frac{6b^4\mu h^3\alpha\beta^3\xi^2}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(k-l+2)(k-l+3)W[k-l+3,p,s]$$
$$+\mu h_o^3\alpha vab^2(k+1)(h+1)(h+2)W[k+1,h+2,m]$$
$$+6\mu h_o^3\alpha\beta vb^2\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$+\frac{9vb^2\mu h_o^3\alpha\beta^2}{a}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$+\frac{12vb^2\mu h_o^3\alpha\beta^2}{a^2}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$+\frac{5vb^2\mu h_o^3\alpha\beta^3}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-4,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$+6vab^2\mu h_o^3\beta(k+1)(h+1)(h+2)W[k+1,h+2,m]$$
$$+12vb^2\mu h_o^3\beta^2\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$+\frac{6vb^2\mu h_o^3\beta^3}{a}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$-6vab^2\mu h_o^3\alpha\beta(k+1)(h+1)(h+2)W[k+1,h+2,m]$$
$$-12vb^2\mu h_o^3\alpha\beta^2\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$-\frac{6vb^2\mu h_o^3\alpha\beta^3\xi^2}{a}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(h-p+1)(h-p+2)W[k-l+1,h-p+2,s]$$
$$+\frac{b^4\mu h^3\alpha}{a^2}(k+1)(k+2)W[k+2,h,m]$$
$$+\frac{9b^4\mu h^3\alpha\beta}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(k-l+2)W[k-l+2,p,s]$$
$$+\frac{18b^4\mu h^3\alpha\beta^2}{a^4}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(k-l+2)W[k-l+2,p,s]$$
$$+\frac{10b^4\mu h^3\alpha\beta^3}{a^5}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(k-l+1)(k-l+2)W[k-l+2,p,s]$$
$$+\frac{3b^4\mu h^3\alpha\beta}{a^2}(k+1)(k+2)W[k+2,h,m]$$
$$+\frac{9b^4\mu h^3\alpha\beta}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(k-l+2)W[k-l+2,p,s]$$
$$+\frac{6\mu h^3\alpha\beta^3b^4}{a^4}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(k-l+1)(k-l+2)W[k-l+2,p,s]$$
$$+\frac{6b^4\mu h^3\beta^2}{a^2}(k+1)(k+2)W[k+2,h,m]$$
$$+\frac{6b^4\mu h^3\beta^3}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(k-l+2)W[k-l+2,p,s]$$

$$
\begin{aligned}
&-\frac{6b^4\mu h^3\alpha\beta^2}{a^2}(k+1)(k+2)W[k+2,h,m]\\
&-\frac{6b^4\mu h^3\alpha\beta^3}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(k-l+1)(k-l+2)W[k-l+2,p,s]\\
&+\mu v b^2 h^3\alpha(h+1)(h+2)W[k,h+2,m]\\
&+\frac{9vb^2\mu h^3\alpha\beta}{a}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(p+1)(p+2)W[k-l,p+2,s]\\
&+\frac{18vb^2\mu h^3\alpha\beta^2}{a^2}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(p+1)(p+2)W[k-l,p+2,s]\\
&+\frac{10vb^2\mu h^3\alpha\beta^3}{a^3}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-3,h-p,m-s)(p+1)(p+2)W[k-l,p+2,s]\\
&+3\mu h^3\alpha\beta v b^2(h+1)(h+2)W[k,h+2,m]\\
&+\frac{9vb^2\mu h^3\alpha\beta^2}{a}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(p+1)(p+2)W[k-l,p+2,s]\\
&+\frac{6vb^2\mu h^3\alpha\beta^3}{a^2}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(p+1)(p+2)W[k-l,p+2,s]\\
&+6vb^2\mu h^3\beta^2(h+1)(h+2)W[k,h+2,m]\\
&+\frac{6vb^2\mu h^3\beta^3}{a}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(p+1)(p+2)W[k-l,p+2,s]\\
&-6vb^2\mu h^3\alpha\beta^2(h+1)(h+2)W[k,h+2,m]\\
&-\frac{6vb^2\mu h^3\alpha\beta^3}{a}\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(p+1)(p+2)W[k-l,p+2,s]\\
&+a^4b^4\rho_o h_o(m+1)(m+2)W[k,h,m+2]\\
&+a^3b^4\rho_o\sigma h_o\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(s+1)(s+2)W[k-l,p,s+2]\\
&+a^3b^4\rho_o h_o\beta\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-1,h-p,m-s)(s+1)(s+2)W[k-l,p,s+2]\\
&+a^2b^4\rho_o h_o\sigma\beta\sum_{l=0}^{k}\sum_{p=0}^{h}\sum_{s=0}^{m}\delta(l-2,h-p,m-s)(s+1)(s+2)W[k-l,p,s+2];
\end{aligned}
\tag{19}
$$

Subsequently, the recursive relation can be re-written as;

$$W\left[k+4,h,m\right]=\frac{\left\{-\left(\begin{array}{l}-\frac{3\left(\left(\beta-1/6\right)\alpha-\beta\right)h_o^3\left(k+1\right)\left(k+2\right)^2b^4\left(k+3\right)\mu W\left[\left[k+3,h,m\right]\right]}{a}\\-\frac{3h_o^3\left(k+1\right)^2\left(k+2\right)^2b^4\left(\left(\beta^2-\beta/2-1/6\right)\alpha-\beta^2\right)\mu W\left[k+2,h,m\right]}{a^2}\\-\frac{h_o^3\beta\left(k+1\right)^2\left(k+2\right)b^4k\left(\left(\alpha-1\right)\beta^2-3/2\alpha\beta-3/2\alpha\right)\mu W\left[\left[k+1,h,m\right]\right]}{a^3}\\+\frac{1}{2}\frac{h_o^3\beta^2\alpha\mu b^4\left(k-1\right)k\left(k+2\right)\left(k+1\right)\left(\beta+3\right)W\left[k,h,m\right]}{a^4}\\+\frac{1}{2}\frac{h_o^3\beta^3\alpha\mu b^4\left(k-1\right)\left(k-2\right)\left(k+2\right)\left(k+1\right)W\left[k-1,h,m\right]}{a^5}\\-2\mu h_o^3a^2b^2\left(h+2\right)\left(h+1\right)\left(k+2\right)\left(k+1\right)\left(\alpha-1\right)W\left[k+2,h+2,m\right]\\-12\left(\left(\beta-1/6\right)\alpha-\beta\right)ah_o^3\left(k+1\right)b^2k\mu W\left[k+1,2,m\right]\\-12b^2k\left(\left(k\beta^2+\left(-k/2-1/2\right)\beta-k/6-1/6\right)\alpha-\beta^2\left(k+1\right)\right)h_o^3\mu W\left[k,2,m\right]\\-\frac{4\left(\left(\alpha-1\right)\beta^2-3/2\alpha\beta-3/2\alpha\right)h_o^3\beta\left(k+1\right)b^2\left(k-1\right)\mu W\left[k-1,2,m\right]}{a}\\+\frac{2h^3\beta^2b^2\mu\alpha\left(\left(k+1\right)\left(k-2\right)\left(\beta+3\right)a+\beta\left(k^2-9\right)\right)W\left[k-2,2,m\right]}{a^3}\\-\mu h_o^3a^4\left(h+4\right)\left(h+3\right)\left(h+2\right)\left(h+1\right)\left(\alpha-1\right)W\left[k,h+4,m\right]\\-3\left(h+2\right)a^3\left(\left(\beta-1/6\right)\alpha-\beta\right)\left(h+4\right)\left(h+1\right)\left(h+3\right)h_o^3\mu W\left[k-1,h+4,m\right]\\-3\left(h+2\right)a^2\left(h+4\right)\left(h+1\right)\left(h+3\right)h_o^3\mu\left(\left(\beta^2-\beta/2-1/6\right)\alpha-\beta^2\right)W\left[k-2,h+4,m\right]\\-\left(h+2\right)a\left(h+4\right)\left(h+1\right)\left(h+3\right)h_o^3\mu\left(\left(\alpha-1\right)\beta^2-3/2\alpha\beta-3/2\alpha\right)\beta W\left[k-3,h+4,m\right]\\+\frac{1}{2}\alpha\beta^2\mu h_o^3\left(h+4\right)\left(h+3\right)\left(n+2\right)\left(h+1\right)\left(\beta+3\right)W\left[k-4,h+4,m\right]\\+\frac{1}{2}\frac{\alpha\mu h_o^3\beta^3\left(h+1\right)\left(h+2\right)\left(h+3\right)\left(h+4\right)W\left[k-5,h+4,m\right]}{a}\\-6ah_o^3\left(\left(\beta-1/6\right)\alpha-\beta\right)\mu\left(k+1\right)\left(h+2\right)b^2\left(h+1\right)W\left[k+1,h+2,m\right]\\-6\left(h+2\right)b^2\left(h+1\right)h_o^3\left(\left(\beta^2-\beta/2-1/6\right)\alpha-\beta^2\right)\mu vW\left[k,h+2,m\right]\\-\frac{6\left(\left(\alpha-1\right)\beta^2-3/2\alpha\beta-3/2\alpha\right)h_o^3v\left(h+1\right)\left(h+2\right)b^2\mu\beta W\left[k-1,h+2,m\right]}{a}\\+\frac{6\beta^2b^2\alpha\mu h_o^3v\left(h+1\right)\left(h+2\right)\left(\beta+3\right)W\left[k-2,h+2,m\right]}{a^2}\\+\frac{10\alpha\beta^3v\mu h_o^3b^2\left(h+1\right)\left(h+2\right)W\left[k-3,h+2,m\right]}{a^3}-a^3b^4h_o\rho\left(m+2\right)\left(m+1\right)\left(\beta+\sigma\right)W\left[k-1,h,m+2\right]\\-a^2b^4\rho h_o\sigma\beta W\left[k-2,h,m+2\right]-a^4b^4h_o\rho\left(m+1\right)\left(m+2\right)W\left[k,h,m+2\right]-Q_o\frac{\omega^m\sin\left(1/2m\pi\right)}{m!}\end{array}\right)\right\}}{-\mu h_o^3b^4\left(k+4\right)\left(k+3\right)\left(k+2\right)\left(k+1\right)\left(\alpha-1\right)};\qquad(20)$$

3. 2. Application of differential transform method to the boundary condition

Applying the principle of DTM stated in Table 1 to the boundary condition Eqs. (10-13)

For $k=0$

$$W[k,n,m]=0 \Rightarrow W[0,n,m]=0,$$
$$(k+1)(k+2)W[k+2,n,m]=0 \Rightarrow W[2,n,m]=0,$$

For $k=1$

$$W[k,n,m]=0 \Rightarrow W[1,n,m]=a_1,$$
$$(k+1)(k+2)W[k+2,n,m]=0 \Rightarrow W[3,n,m]=\frac{b_1}{6}, \tag{21}$$

The order of the derivative of the governing equation determines the number of unknowns required for the analysis. Therefore, four unknowns are required for this analysis, two given and others are assigned as unknown.

$$\begin{aligned} W[0,n,m]&=0,\\ W[1,n,m]&=a_1,\\ W[2,n,m]&=0,\\ W[3,n,m]&=\frac{b_1}{6}, \end{aligned} \tag{22}$$

where a_1 *and* b_1 are unknown constant to be found later.

3. 3. The solution procedure

By fixing k and varying h,m and vice versa $[k,h,m]=0,1,2,3,4,5,6,7\cdots\cdots$, in Eq. (20), the following are developed;

$$W_{4,0,0}=-\frac{w_o\begin{pmatrix}22664835170\alpha\beta^2+709821428600\alpha\beta-22664835170\beta^2+\\ 35590315940000\alpha-721153846200\beta-35714285720000\end{pmatrix}}{2060439560581\alpha-2060439560581} \tag{23}$$

$$W_{5,0,0}=-\frac{\left\{\begin{matrix}-181318681400\alpha^2\beta^3w_o-4925480769000\alpha^2\beta^2w_o+362637362700\alpha\beta^3w_o-\\ 204834993100000\alpha^2\beta w_o+9540865384000w_o\alpha\beta^2-181318681400\beta^3w_o+\\ 445930116900000\alpha^2w_o+410861950500000w\ \alpha\beta-4615384615000w_o\beta^2+\\ 3000000000000a_1\alpha-858203983700000w_o\alpha-206043956000000w_o\beta-\\ 3000000000000a_1+412087912200000w_o\end{matrix}\right\}}{103021978018440(\alpha-1)^2}; \tag{24}$$

$$W_{6,0,0}=\frac{\left\{\begin{array}{l}-1699862638000\alpha^{3}\beta^{4}w_{o}-44833447800000\alpha^{3}\beta^{3}w_{o}+5099587914000\alpha^{2}\beta^{4}w_{o}-\\1761843234000000\alpha^{3}\beta^{2}w_{o}+129330357100000\alpha^{2}\beta^{3}w_{o}-5099587914000\alpha\beta^{4}w_{o}+\\3155613411000000\alpha^{3}\beta w_{o}+5320134786000000\alpha^{2}\beta^{2}w_{o}-124160370900000\alpha\beta^{3}w_{o}+\\1699862638000\beta^{4}w_{o}+37500000000000a_{1}\alpha^{2}\beta-75000000000000a_{1}\alpha^{2}\sigma-\\9004863928000000\alpha^{3}w_{o}-4120879122000000\alpha^{3}W_{4,2,0}-9095715147000000\alpha^{2}\beta w_{o}-\\5353964627000000w_{o}\alpha\beta^{2}+39663461530000\beta^{3}w_{o}-75000000000000a_{1}\alpha^{2}-\\75000000000000a_{1}\alpha\beta+150000000000000a_{1}\alpha\sigma+26168275250000000\alpha^{2}w_{o}+\\12362637370000000\alpha^{2}W_{4,2,0}+8721737639000000w_{o}\alpha\beta+1795673076000000w_{o}\beta^{2}+\\75000000000000a_{1}\alpha+37500000000000a_{1}\beta-75000000000000a_{1}\sigma-25404704680000000w_{o}\alpha\\-12362637370000000\alpha W_{4,2,0}-2781593408000000w_{o}\beta+8241758244000000w_{o}+\\4120879122000000W_{4,2,0}\end{array}\right\}}{7726648352975042(\alpha-1)^{3}}; \qquad (25)$$

$$W_{4,1,0}=-\frac{w_{o}\left(\begin{array}{l}2961881868\alpha\beta^{2}+95102163520\alpha\beta-2961881868\beta^{2}+\\10800716860000\alpha-96583104450\beta-10817307690000\end{array}\right)}{128777472533\alpha-128777472533}; \qquad (26)$$

$$W_{5,1,0}=-\frac{\left\{\begin{array}{l}-18956043950000000000\alpha^{2}\beta^{3}w_{o}-59922733530000000000\alpha^{2}\beta^{2}w_{o}\\+37912087890000000000\alpha\beta^{3}w_{o}-513340229600000000000\alpha^{2}\beta w_{o}\\+11885130500000000000w_{o}\alpha\beta^{2}-18956043950000000000\beta^{3}w_{o}\\+1033277902000000000000\alpha^{2}w_{o}+1028432348000000000000w_{o}\alpha\beta\\-58928571440000000000w_{o}\beta^{2}+150000000000000000000a_{1}\alpha\\-1981279190000000000000w_{o}\alpha-515109889500000000000w_{o}\beta\\-150000000000000000000a_{1}+947802198000000000000w_{o}\end{array}\right\}}{5151098901983361762016(\alpha-1)^{2}}; \qquad (27)$$

$$W_{6,1,0}=-\frac{\left\{\begin{array}{l}1777129120000\alpha^3\beta^4w_o-57402987640000\alpha^3\beta^3w_o+5331387359000\alpha^2\beta^4w_o\\-445472827300000\alpha^3\beta^2w_o+170180288500000\alpha^2\beta^3w_o-5331387359000\alpha\beta^4w_o\\+820816470700000\alpha^3\beta w_o+1339579711000000\alpha^2\beta^2w_o-168151614100000\alpha\beta^3w_o\\+1777129120000\beta^4w_o+187500000000000a_1\alpha^2\beta-3750000000000a_1\alpha^2\sigma\\-2256769038000000\alpha^3w_o-6181318683000000\alpha^3W_{4,3,0}-2368366973000000\alpha^2\beta w_o\\-1342716087000000w_o\alpha\beta^2+55374313180000\beta^3w_o-3750000000000a_1\alpha^2\\-37500000000000a_1\alpha\beta+7500000000000a_1\alpha\sigma+6552758029000000\alpha^2w_o\\+1854395605000000\alpha^2W_{4,3,0}+2273859891000000w_o\alpha\beta+448609202500000w_o\beta^2\\+3750000000000a_1\alpha+18750000000000a_1\beta-3750000000000a_1\sigma\\-6356378779000000w_o\alpha-1854395605000000\alpha W_{4,3,0}-726304945200000w_o\beta\\+2060439561000000w_o+6181318683000000W_{4,3,0}\end{array}\right\}}{3863324176487521(\alpha-1)^3};\quad(28)$$

$$W_{4,2,0}=-\frac{w_o\left(\begin{array}{l}4223901102\alpha\beta^2+136967719800\alpha\beta-4223901102\beta^2+\\25525566630000\alpha-139079670400\beta-25549450560000\end{array}\right)}{103021978018\alpha-103021978018};\quad(29)$$

$$W_{5,2,0}=\frac{-\left\{\begin{array}{l}-3379120880000000000\alpha^2\beta^3w_o-112865728000000000000\alpha^2\beta^2w_o\\+6758241761000000000\alpha\beta^3w_o-1525328726000000000000\alpha^2\beta w_o\\+225571772000000000000w_o\alpha\beta^2-3379120880000000000\beta^3w_o\\+2974021893000000000000\alpha^2w_o+3054143201000000000000w_o\alpha\beta\\-112706044000000000000w_o\beta^2+150000000000000000000a_1\alpha\\-5694160372000000000000w_o\alpha-1528846154000000000000w_o\beta\\-150000000000000000000a_1+2719780220000000000000w_o\end{array}\right\}}{515109890198336176216(\alpha-1)^2};\quad(30)$$

$$W_{4,3,0}=-\frac{w_o\left(\begin{array}{l}2232142858\alpha\beta^2+72716346170\alpha\beta-2232142858\beta^2+\\19675967270000\alpha-73832417600\beta-19688644700000\end{array}\right)}{34340659340\alpha-34340659340};\quad(31)$$

$$W_{5,3,0} = -\frac{\left\{\begin{array}{l} -53571428570000000000\alpha^2\beta^3 w_o - 1834563873000000000000\alpha^2\beta^2 w_o \\ +10714285710000000000\alpha\beta^3 w_o - 35339873790000000000000\alpha^2\beta w_o \\ +3678657280000000000000 w_o\alpha\beta^2 - 5357142857000000000\beta^3 w_o \\ +67704717560000000000000\alpha^2 w_o + 70737723220000000000000 w_o\alpha\beta \\ -1844093406000000000000 w_o\beta^2 + 150000000000000000000 a_1\alpha \\ -129523609200000000000000 w_o\alpha - 35398351650000000000000 w_o\beta \\ -150000000000000000000 a_1 + 61813186830000000000000 w_o \end{array}\right\}}{5151098901983361762016(\alpha-1)^2}; \quad (32)$$

$$W_{6,3,0} = \frac{\left\{\begin{array}{l} -2008928570000000000\alpha^3\beta^4 w_o - 71941964220000000000\alpha^3\beta^3 w_o \\ +6026785708000000000\alpha^2\beta^4 w_o - 1232368216000000000000\alpha^3\beta^2 w_o \\ +215793269100000000000\alpha^2\beta^3 w_o - 6026785708000000000\alpha\beta^4 w_o \\ +2166466818000000000000\alpha^3\beta w_o + 3700372507000000000000\alpha^2\beta^2 w_o \\ -215760645500000000000\alpha\beta^3 w_o + 2008928570000000000\beta^4 w_o \\ +75000000000000000000 a_1\alpha^2\beta - 150000000000000000000 a_1\alpha^2\sigma \\ -6293382257000000000000\alpha^3 w_o - 824175824400000000000\alpha^3 W_{4,5,0} \\ -6237349499000000000000\alpha^2\beta w_o - 3703649894000000000000 w_o\alpha\beta^2 \\ +71909340590000000000\beta^3 w_o - 150000000000000000000 a_1\alpha^2 \\ -150000000000000000000 a_1\alpha\beta + 300000000000000000000 a_1\alpha\sigma \\ +18297251120000000000000\alpha^2 w_o + 2472527473000000000000\alpha^2 W_{4,5,0} \\ +5974733858000000000000 w_o\alpha\beta + 1235645603000000000000 w_o\beta^2 \\ +150000000000000000000 a_1\alpha + 75000000000000000000 a_1\beta - 150000000000000000000 a_1\sigma \\ -17773042590000000000000 w_o\alpha - 2472527473000000000000\alpha W_{4,5,0} \\ -1903846154000000000000 w_o\beta + 5769230772000000000000 w_o \\ +824175824400000000000 W_{4,5,0} \end{array}\right\}}{15453296703562041495596(\alpha-1)^3}; \quad (33)$$

$$W_{4,4,0} = -\frac{w_o\left(\begin{array}{l} 19574175830\alpha\beta^2 + 639251374100\alpha\beta - 19574175830\beta^2 + \\ 234778674500000\alpha - 649038462000\beta - 234890109900000 \end{array}\right)}{206043956058\alpha - 206043956058}; \quad (34)$$

$$W_{5,4,0}=-\frac{\left\{\begin{array}{l}-52197802210\alpha^2\beta^3 w_o-1811298078000\alpha^2\beta^2 w_o+104395604400\alpha\beta^3 w_o\\-468921445900000\alpha^2\beta w_o+3638221157000w_o\alpha\beta^2-52197802210\beta^3 w_o\\+888614726400000\alpha^2 w_o+938422046800000w_o\alpha\beta-1826923078000w_o\beta^2\\+100000000000a_1\alpha-1699110004000000w_o\alpha-469505494500000w_o\beta\\-100000000000a_1+810439560400000w_o\end{array}\right\}}{3434065934065(\alpha-1)^2}; \quad (35)$$

$$W_{4,5,0}=-\frac{w_o\left(\begin{array}{l}13495879120\alpha\beta^2+441397665000\alpha\beta-13495879120\beta^2+\\210499982900000\alpha-448145604600\beta-210576923100000\end{array}\right)}{103021978018\alpha-103021978018}; \quad (36)$$

Using the definition of DTM;

$W[x,y,t]=\sum_{i=0}^{6}\sum_{u=0}^{5}\sum_{j=0}^{5}W[i,u,j]x^i y^u t^j$, the solution of governing differential equation (8) becomes to find the unknow introduced into the boundary conditions, substitute $x=1, y=0.0045, t=0.003$ into Eq (12), we have

$$-\frac{1}{7447315263997704177}-\frac{381b_1}{2187823}+\frac{56409a_1}{24872}; \quad (37)$$

Substitute $x=1, y=0.0045, t=0.003$ into Eq. (13), one obtained

$$-\frac{95b_1}{5040037}+\frac{82624a_1}{136041}+\frac{1}{7651241724074558173380}; \quad (38)$$

Solving equation Eqs. [(37) and (38)]

$$a_1=-\frac{1}{2284411039920108736\,4},\; b_1=-\frac{1}{745624173679718} \quad (39)$$

Putting Eq. (39) into Eq. (36), the series solution of Eq. (8) is obtained as

$$\begin{aligned}w(x,y) = &-\frac{x}{5711027599147955116}-\frac{x^6y^5}{369312062611075}-\frac{x^6y^4}{518278273005307}-\frac{x^6y}{2681526718098306}\\&-\frac{x^6y^2}{1312236478425518}-\frac{x^6y^3}{780697652141669}+\frac{x^4y^5}{4111674300937062}+\frac{x^4y}{4111674300937062}+\frac{x^4y^4}{4111674300937062}\\&+\frac{x^4y^2}{4111674299246475}+\frac{x^4y^3}{4111674300937062}-\frac{xy^5}{5711027599147955116}-\frac{xy^2}{5711027599147955116}\\&-\frac{xy^3}{5711027599147955116}-\frac{xy^4}{5711027599147955116}-\frac{xy}{5711027599147955116}+\frac{x^5y^4}{92446571179047}+\frac{x^5y}{455119448504566}\\&+\frac{x^5y^3}{138484481466813}+\frac{x^5y^5}{660495739515 82}-\frac{x^3y}{1118436260561273}-\frac{x^3y^4}{1118436260561273}+\frac{x^5y^2}{229954744860220}\\&-\frac{x^3y^3}{1118436260561273}-\frac{x^3y^2}{1118436260561273}-\frac{x^3y^5}{1118436260561273}-\frac{x^3}{1118436260561273}+\frac{x^5}{1308189389239784}\\&+\frac{x^4}{4111674300937062}-\frac{x^6}{8810730643830079};\end{aligned} \tag{40}$$

4. RESULTS AND DISCUSSION

DTM solution of the governing differential equation is presented herein, the rectangular plate is supported at one side as simply-support and on the other side as free edge support and subjected to uniform distributed load $Q(x,y) = \mathrm{Q}_o$. The maximum deflection in x and y direction is denoted as $W_{\max}$.

The fundamental natural frequency converged at eight iterations using DTM. The maximum deflection of the rectangular plate is obtained using maximum deflection which is obtained using $W_{\max} = W/\left(Q_o a^4 / D_o\right)$ [25]: where W is deflection, Q is external loading, μ is flexural rigidity and $W_{\max}$ is maximum deflection and is determined for different values of fundamental natural frequency $\omega = 0.3\omega_o, 0.5\omega_o$ and $0.8\omega_o$ respectively.

The maximum dynamic response of plates under thermal influence is obtained by the present method and confirmed in good agreement with [26] for all values of the controlling parameters choosen.

4. 1. Effect of taper constant *β*

In order to investigate the influence of the controlling parameters on the maximum deflection, different values of *α, β, λ* are chosen as listed in the table of results. The maximum deflection obtained for different values of taper constant are listed in Table 3. From the result, it is depicted that as the homogenous constant, which is a function of density increases, the rectangular plate maximum deflection increases, as shown in Fig. 2 and the difference is more pronounced at the higher value of the taper constant.

This means that, increase in density results in rectangular plate flexibility increase thereby increase the maximum deflection. The parameter may be used as guide to control the deflection of plates.

Table 3. Showing Taper Constant variation

Edge Condition	Taper Constant	Maximum deflection		
		α = 0, initial thickness (h_o = 0.05), initial density (ρ = 8000), Q_o = 1.5E-9		
	β	$\omega = 0.3\omega_o$	$\omega = 0.5\omega_o$	$\omega = 0.8\omega_o$
Simply Free	0	-1.16E-08	-6.18E-08	-2.65E-07
	0.2	-1.12E-08	-5.99E-08	-2.57E-07
	0.4	-1.08E-08	-5.99E-08	-2.48E-07
	0.6	-1.05E-08	-5.61E-08	-2.40E-07
	0.8	-1.01E-08	-5.42E-08	-2.32E-07
	1	-9.78E-09	-5.23E-08	-2.24E-07

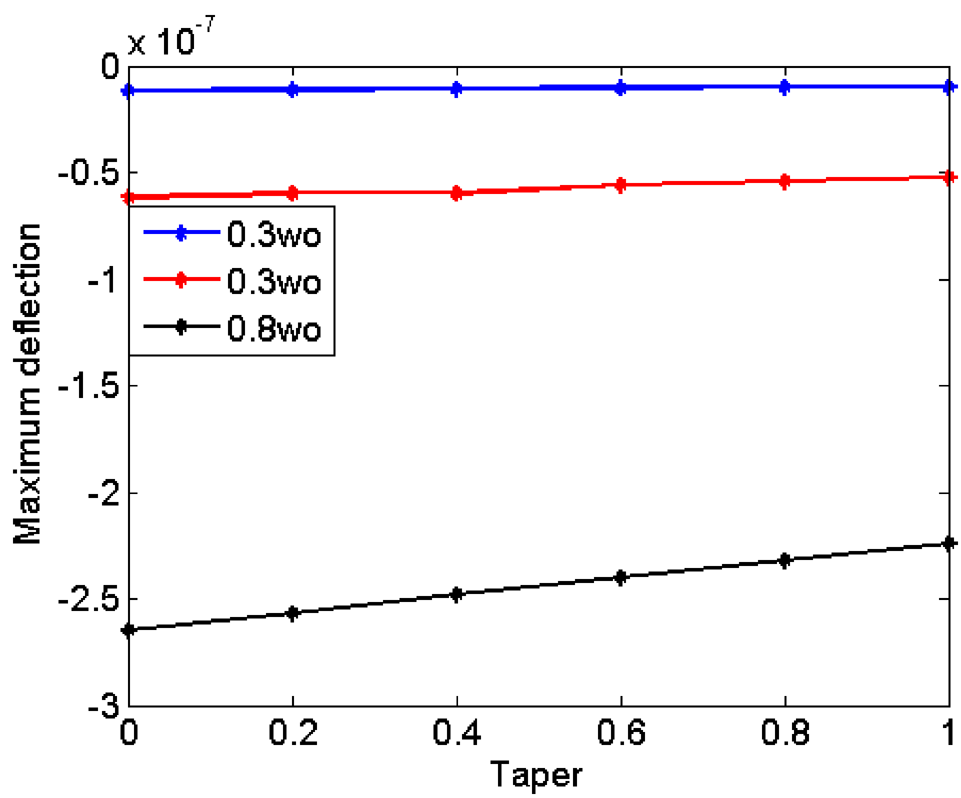

Fig. 2. Taper variation on maximum deflection

4. 2. Effect of thermal constant

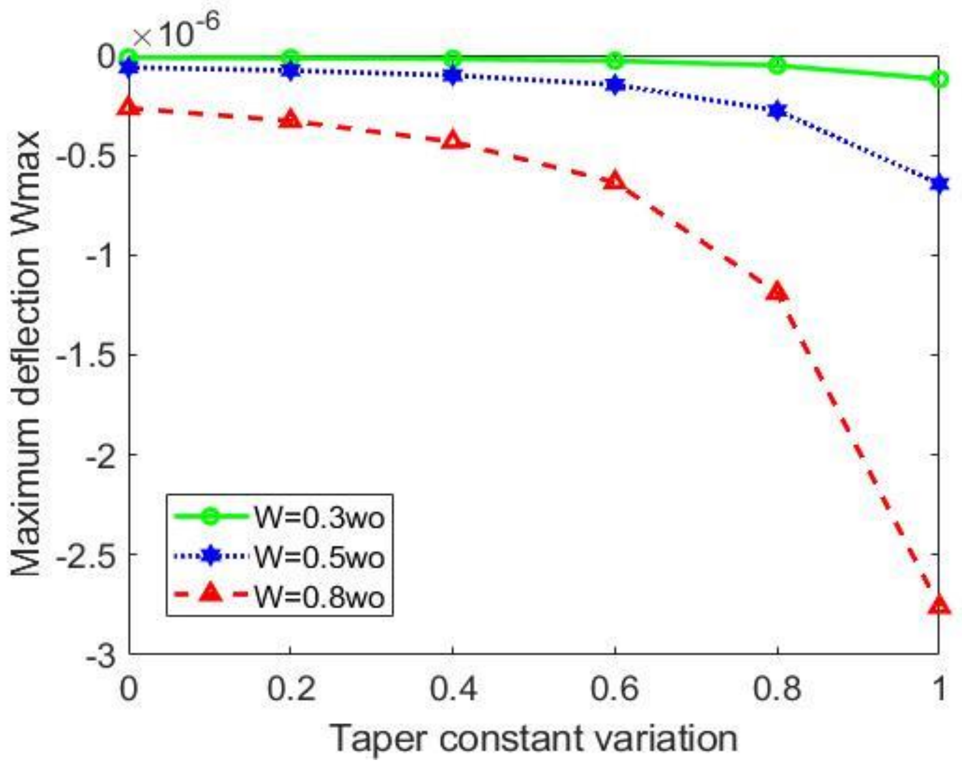

Fig 3. Showing thermal constant variation

Table 4. Showing thermal constant variation.

Edge Condition	Thermal Constant	Maximum deflection		
		$\beta = 0$, initial thickness ($h_o = 0.05$), initial density ($\rho = 8000$) , $Q_o = 1.5E\text{-}9$		
	α	$\omega = 0.3\omega_o$	$\omega = 0.5\omega_o$	$\omega = 0.8\omega_o$
Simply Free	0	-1.16E-08	-6.18E-08	-2.65E-07
	0.2	-1.43E-08	-7.67E-08	-3.29E-07
	0.4	-1.89E-08	-1.01E-07	-4.33E-07
	0.6	-2.78E-08	-1.48E-07	-6.36E-07
	0.8	-5.18E-08	-2.77E-07	-1.19E-06
	1	-1.20E-07	-6.44E-07	-2.76E-06

The influence of temperature changes is illustrated in Table 4 and Fig. 3. It is shown that, increase in thermal parameter leads to decrease in the maximum deflection of the rectangular plate. As the thermal parameters increases, the bending stiffness and the Young's modulus which are the determinant of temperature in the model varies. At low temperature, the Young's modulus and bending stiffness decrease at low temperature as a result, the rectangular plate becomes flexible thereby results in increase in maximum deflection. However, at higher temperature, increase in temperature change results in decrease in maximum deflection of the rectangular plate. This is because the Young modulus and flexural rigidity of rectangular plate, a function of temperature increase at high temperature and resulting in rectangular plate becomes increasingly rigid as the temperature change increase, thereby resulting into decrease in maximum deflection of the vibrating rectangular plate. The thread of the response is illustrated in Fig. 3. Invariably, the thermal conductivity increases with the presence of heat flux.

4. 3. Effect of aspect ratio a/b

Fig. 4 shows that increases in aspect ratio leads to decreases in maximum deflection. Aspect ratio in this case implies the dimension of the plate in x and y direction. Increasing the dimension increases the surface area, thereby increases the mass and mass is inversely proportion to deflection in vibration.

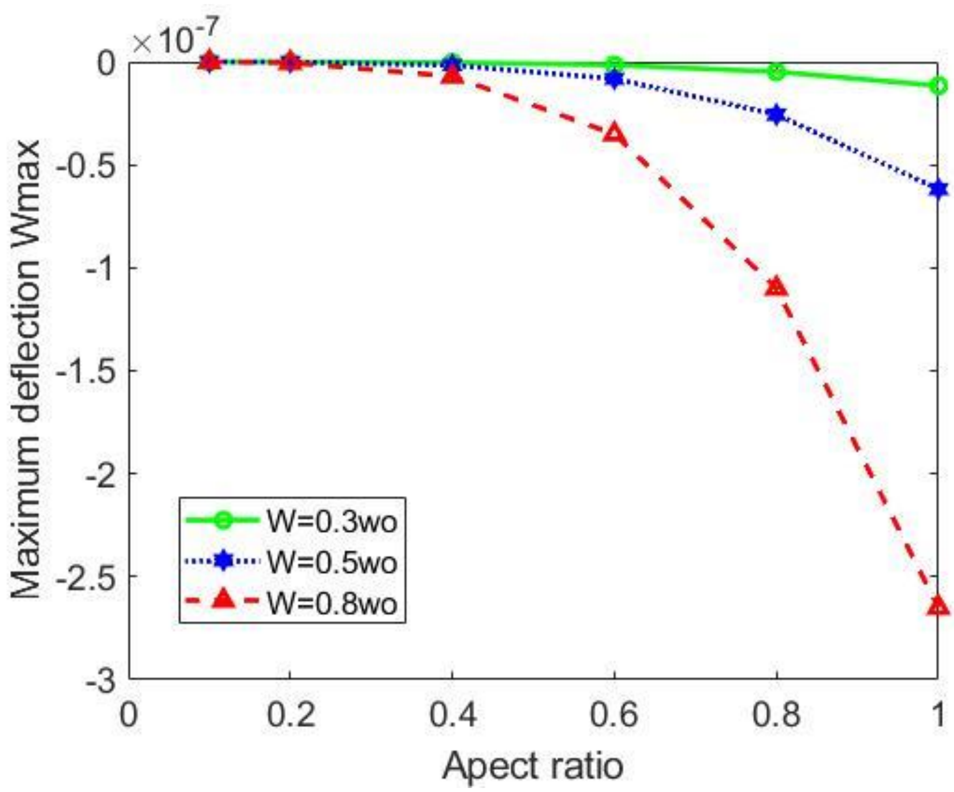

Fig. 4. Showing aspect ratio variation

From the result obtained, it is observed that, as the aspect ratio increase, the maximum deflection decreases, this shows that increase in size of the rectangular plate leads to rectangular plate becoming more rigid and consequently resulting into decrease in maximum deflection. The decrease in more pronounce as the higher value of the aspect ratio.

Table 5. Showing Aspect ratio variation.

Edge Condition	Aspect ratio	Maximum deflection		
		$\beta = 0$, $\alpha = 0$, ($h_o = 0.05$), ($\rho = 8000$), $Q_o = 1.5E\text{-}9$		
	λ	$\omega = 0.3\omega_o$	$\omega = 0.5\omega_o$	$\omega = 0.8\omega_o$
Simply Free	0.1	-1.25E-12	-6.60E-12	-2.82E-11
	0.2	-1.99E-11	-1.05E-10	-4.48E-10
	0.4	-3.12E-10	-1.65E-09	-7.05E-09
	0.6	-1.55E-09	-8.21E-09	-3.51E-08
	0.8	-4.80E-09	-2.56E-08	-1.10E-07
	1	-1.16E-08	-6.18E-08	-2.65E-07

4. 4. Effect of natural frequency ω_o

From Figs. 2-4, it is shown that, increase in natural frequency correspond to increases in maximum deflection. Natural frequency is the inertial energy of the system, increasing the inertial energy increases the natural frequency and deflection respectively.

Table 6. Showing natural frequency variation.

Edge Condition	Aspect ratio	Maximum deflection		
		($h_o = 0.05$), ($\rho = 8000$), $Q_o = 1.5E\text{-}9$		
	Parameters	$\omega = 0.3\omega_o$	$\omega = 0.5\omega_o$	$\omega = 0.8\omega_o$
Simply Free	$\lambda = 1$, $\beta = 0.2$, $\alpha = 0.1$,	-1.20E-08	-6.63E-08	-2.84E-07
	$\lambda = 2$, $\beta = 0.5$, $\alpha = 0.7$	-3.00E-07	-1.65E-06	-7.09E-06
	$\lambda = 2$, $\beta = 0.5$, $\alpha = 0.7$	-4.16E-06	-2.24E-05	-9.63E-05

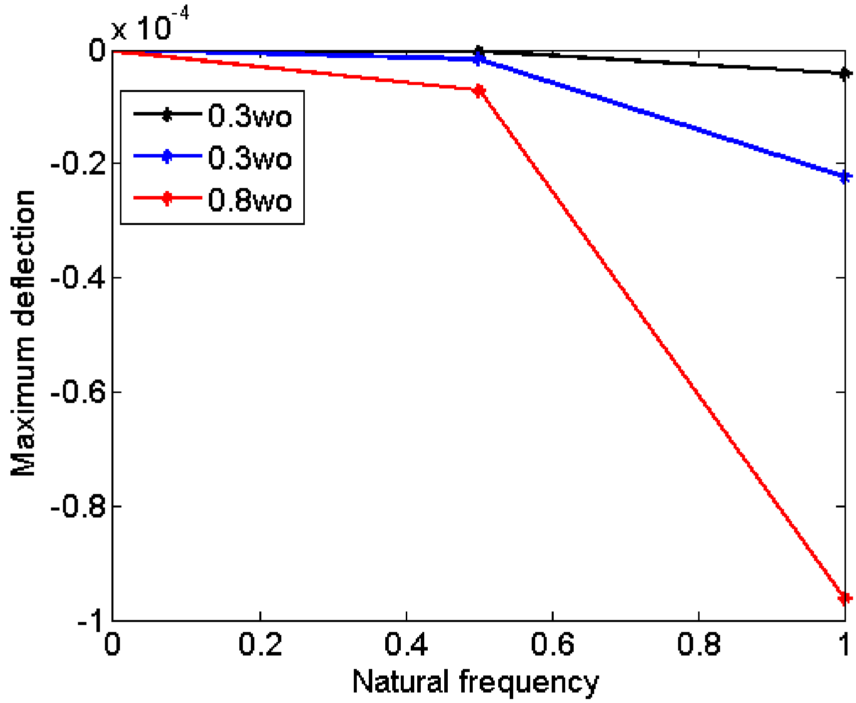

Fig. 5. Natural frequency on maximum deflection

4. 5. Effect of non-homogeneity

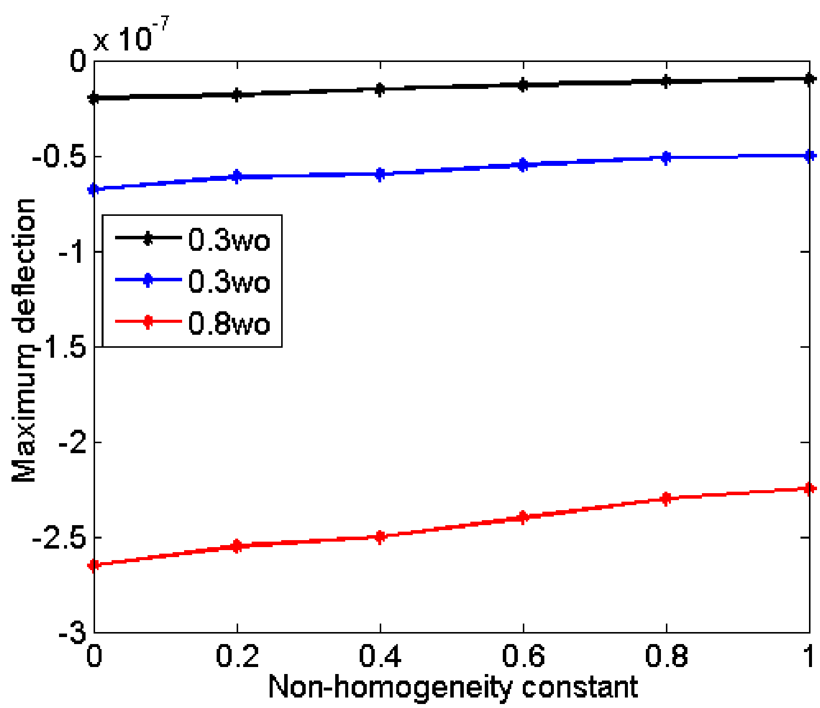

Fig. 6. Showing variation of non-homogeneity

Fig. 6 shows that non-homogeneity constant increases with increase in maximum deflection. From the results, it is observed that as homogeneity which is a function of density variation increases the maximum deflection increases. The rectangular plate flexibility increases with increase in homogeneity thereby resulting into increase maximum deflection of the rectangular plate.

5. CONCLUSIONS

Nonlinear thermally induced dynamic analysis of non-homogenous rectangular plate with varying thickness in presence of external excitation have been analyzed using Three-dimensional differential transform method. The analytical solution obtained used to obtain the maximum deflection of the plate which is later used for determining the effect of taper constant, aspect ratio, thermal constant and non-homogeneity. Based on the findings, the underlisted are obtained;

I. Taper constant increases with increase in maximum deflection.
II. Thermal constant increases with decreases in maximum deflection.
III. Increases in aspect ratio leads to decreases in maximum deflection
IV. Increase in natural frequency correspond to increases in maximum deflection.
V. Non-homogeneity constant increases with increase in maximum deflection.

The effect of taper constant, thermal effect, aspect ratio and natural frequency on maximum deflection of rectangular plate is hereby shown after the study.

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