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Complexification of operational infinity

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ABSTRACT

It is demonstrated here by examples that infinitesimal descending infinity is formwise analogous to the Cauchy integral formula. Hence the concept of total, twin operational infinity is represented by complex formula combining the real neverending ascending infinity and the imaginary infinitesimal descending infinity, each depicted in separate abstract spaces that appear as dual reciprocal, though.

***Keywords*:** Complexified descending infinity, division by zero, multiplication by infinity

1. INTRODUCTION

Traditional mathematics implicitly relied on single space reality (SSR) paradigm in the context of which geometric spaces had been identified with sets that (set-theoretically) can be viewed as mere selections from a single universal superset. The SSR paradigm inadvertently compromised some operationally legitimate differential procedures, which in turn effectively concealed the necessity of presence of certain abstract quasigeometric structures that could correspond to legitimate differential procedures, if these had been recognized.

Therefore, the SSR paradigm should be replaced with a certain multispatial reality (MSR) paradigm that supports the formerly delegitimized operational procedures, presumably made in the past as a shoelicking attempt to stay in line with the faulty traditional ways of doing mathematics. Under auspices of the MSR paradigm, however, some traditional interpretations

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of well-known and previously proven mathematical theorems and even some informal concepts can acquire formerly unanticipated extensions. Operational infinity is one of them.

While the infinitesimal descending infinity frequently encountered in complex analysis is usually detoured, the neverending ascending infinity that pops up in many applications of real analysis, remains controversial and thus avoided whenever possible. I want to show by examples that these twin concepts can be interpreted as two faces of the same coin, namely complexified operational infinity. The prerequisites for this topic are [1-3].

Conceptual importance of infinity in mathematics is not restricted to providing meaningful division by zero, which I have already proposed to be performed as multiplication by infinity, for it also mandates corrections to some differential operators indispensable in physical sciences. I must use reasonings by examples because the topics to be discussed in this paper cannot be derived from axiomatics of traditional mathematics and thus cannot be proven at this present stage of theoretical development of the topics.

Besides, the scope of validity of proofs depends on the paradigms espoused by the prover, some of which may seem to be so selfevident that they are often unmentioned. As paradigms tend to shift, however, the scope of validity of former proofs can shrink and thus some of their generalizations may even become inadmissible. The pathetic history of the notion of parallels in Euclidean geometry is an example of many futile attempts to prove the improvable and disheartening struggles to defend the indefensible, which amounted to fighting shadows for well over twenty centuries. Proofs indicate consistency of derivations, not really the validity of the proven theorems. And yet that trend to justify the often arbitrary postulative concepts with the help of shaky proofs relying on fleeting paradigms continues undaunted.

2. MAPPING INFINITY VIA STEREOGRAPHIC PROJECTION REVEALS DIVERSE NATURE OF INFINITY

Although many authors, and especially those who espoused the idea of unconventional division by zero [4-6], virtually presumed that just because they have implemented their unconventional division by zero in the complex-analytic domain of complex numbers, their definition of division by zero somehow inherited the infinity by default, mainly because of mapping of Riemann sphere the infinity seems to be inherited too. But this expectation that operationally meaningful infinity is indeed included in complex analysis, was an unjustified exaggeration (on the part of Riemann and of some of his followers) perpetuated in virtually all textbooks and papers on complex analysis and related topics.

For infinity in complex-analytic framework is just a transient entity to be detoured like a skunk. It appears on paper and in the minds of the mathematicians who use it as tendency or limit in diagrams, but it cannot be really located anywhere in the realistic space of motion in which physical experiments take place.

This issue intimidates the general theory of relativity (GTR) where the problem is called nonlocalization of energy. Therefore, the question of what the operational infinity is, and where it actually dwells, pertain not only to theoretical mathematics but is also of utmost significance to physics. Hence the concept of infinity should not be treated with almost statutory disregard for physics as it was often done in the past.

Assuming that we live in a single space often identified with set – under then unspoken and thus unquestioned SSR paradigm – Riemann adopted stereographic projection from

Ptolemy [7] and in doing so he somewhat justified his inability to grasp the extra reality that is veiled beyond the SSR-induced reality and thus virtually hid the, extraneous to him, reality standing behind the visible one. Nevertheless, his visualization of infinity via mapping of sphere spurred investigations of dimensionality, which is his great inspirational achievement.

Yet because Riemann assumed that dimensions can be added arbitrarily and stacked upon one another like bricks, just as Grassmann did, today the conceptual impact of his once ingenious ideas became more detrimental than motivating. Grassmann, and then Clifford, algebraized geometric dimensionality, but they did not really succeed in associating it with differentiation, which feat Riemann accomplished. However, the North pole of the Riemann's sphere cannot be [unambiguously] mapped onto equatorial plane – see [8], [9] p.60, even though some authors depict it as if indeed it were possible [10, 11]. For adding a point at infinity see [12]; it apparently makes infinity an honorary member of complex numbers [13] and "Convergence to ∞ refers to a very special kind of divergence." see [14] p.209, which seems only to deepen the confusion that traditionally surrounded the concept of infinity.

Having said that I should mention that Elie Cartan further generalized their efforts and merged the geometric and algebraic trends into the method of exterior differential forms, which shall be discussed elsewhere, because review of the latter requires a reevaluation of Riccci's tensor calculus for the discussion of exterior forms to be really enlightening.

Some authors argue that singularities, which could harbor infinity, should be detoured [15]; see also the subsection discussing integrals around singularities [16]. Notice the word 'around' instead of the phrase "emerging from within", which one could rightly expect. Others advise that expressions like $\infty+\infty$, $\infty-\infty$, and $\infty\cdot 0$ should be avoided for they allegedly can occur only as abbreviations for limiting processes, see otherwise excellent textbook [17]. Although this warning can surely be appreciated, we shall remember that mathematics and physical sciences created truly predictive, no-nonsense theories when Newton invented, and Leibniz simplified, differential calculus whose essence relies on estimating theoretical limits of previously difficult to discern mathematical expressions [18]. Note that classical approach to handling of singularities in some nonlinear partial differential equations, developed for the use in practical applications, was thoroughly exemplified in [19].

My preliminary conclusion from these honest but conceptually somewhat misguided advices, is that the concept of infinity is not quite well understood. The infinity is still neither an operational entity nor truly structurally meaningful geometrical object even though both the operational and structural concepts of infinity are of crucial importance for exact sciences.

Although infinity can be introduced either as a limit or by [postulative] convention [20] it was often treated as just a crutch [21, 22], and blatantly mishandled in the past, creating many embarrassing yet tacitly concealed operational nonsenses, some of which have been exposed in [23]. The previously concealed nonsenses are not really due to the allegedly paradoxical nature of infinity but rather stem from the postulative character of quite arbitrary, yet often uncontested, former assumptions of the traditional mathematics about the infinity.

Although operational infinity is setvalued variable, at least in the set-theoretical sense of the term 'infinity', it is known that in every sufficiently small punctured neighborhood of an isolated essential singularity of a function f(), the range of f() omits at most one point [24].

If so, then perhaps the omitted single point should be regarded as an abstract gateway to yet another space in which the singularity could be treated as truly setvalued operational entity.

Thus, the shift from the SSR to the MSR paradigm seems implicit in the complex analysis.

3. COMPLEX ANALYSIS IMPLIES MULTISPATIAL CHARACTER OF DESCENDING INFINITY

From Green's formula written in complex form [25] the Cauchy's integral theorem (CIT) follows: $\oint_C f(z)\,dz = 0$ [26], [27], [28], [29]. The conceptual meaning of Cauchy's integral formula (CIF) [30] as a reciprocal counterpart of the CIT, can also be viewed as hint at the necessity of making shift (highlighted in red) from the SSR paradigm to the MSR paradigm

$$\oint_C \frac{f(z)dz}{z-z_0} = \oint_{(C-z_0)} \frac{f(z)dz}{z-z_0} + \oint_{C(z_0)} \frac{f(z)dz}{z-z_0} \Longrightarrow \{SSR\} \oslash \{MSR1 \oplus \ldots \oplus MSRn\} \tag{1}$$

for it means separation of the contour C enclosing the singular point/pole z_0. There $(C- z_0)$ is the contour without the point z_0, and $C(z_0)$ is the extracted and separated contour about the point z_0 alone; see [31-36], [37] p.91ff, or illustrated in [38]. Compare also various examples given in [39]. As usual, f(z) denotes a function of the complex variable z. The right-hand side (RHS) of the implication (1) is the reinterpretation of the CIF under the MSR paradigm. The MSRn in (1) denotes the n^{th} separate singular space integral.

The CIF effectively demands separation of the primary domain within the contour C from the extracted domain surrounding the singular point z_0. CIF, as the counterpart of the CIT with reciprocal point singularity virtually demands separation of the singular point. The nonsingular contour corresponds to the SSR paradigm whereas the MSR paradigm covers the extracted contour enclosing the singular points turned into paired dual reciprocal spaces in the MSR setting. There the interspatial addition sign $\oplus$ means additive composition whereas the interspatial subtraction sign $\oslash$ means there that the singular MSR part of CIF is still evaluated in the SSR setting. Thus, the need to shift the SSR paradigm to the MSR paradigm is implicit in the CIF. Compare also the Poisson integral formula, which is defined on unit disk (0,1) in [40]. Recall that in the analogous Cauchy-Goursat theorem: $\oint_{C=\partial O} f(z)\,dz = 0$ the contour C is also the boundary ∂O of the rectangle O, compare [41].

The traditional complex-analytic formulation of the CIF remains unchanged under the MSR paradigm but the newly acquired reinterpretation of its prospective multispatial meaning is far more comprehensive, for in the MSR framework the complex analysis appears as a simplified theory of multispatial geometric structures squeezed into essentially planar 2D algebraic setting, which is clearly incomplete in 3D or in higher-dimensional framework.

The MSR paradigm does not defy complex analysis but makes it viable both operationally and – especially – structurally. Thus, if the separation facilitated by the CIF would be abstracted into a higher-dimensional hypercomplex domain then it could mean associating (or pairing in the new MSR parlance) the secondary dual reciprocal spaces with the primary hypercomplex space, in which case the imaginary unit i becomes an algebraic interspatial operator to be denoted here by î, or the geometric multispatial operator $\vec{\imath}$ whose imaginary value is equal: $|\hat{\imath}| = |\vec{\imath}| = |i| = \sqrt{-1}$. This imaginary symbol î is used in an algebraic/operational context whereas $\vec{\imath}$ can be applied in a geometric/structural context, respectively.

In order to pursue and implement the hint implicit in the CIF a conceptual shift from the SSR paradigm to the MSR paradigm is necessary also for the sake of maintaining algebraic consistency of operations. Recall that continuous deformation of the contour does not affect or alter the integral if the singularity of the integrand is not crossed [42], which fact also demands a separate space for the singularity to dwell in, in compliance with fundamental tenets of the MSR paradigm. For CIT is inapplicable when the contour encloses singularities of the function f(z) [43], which fact also hints at feasibility of the multispatial approach that is offered here. For an intuitive approach to CIT and contour deformations (or homotopy) see [44-48], or an other standard textbook of complex analysis.

In the traditional SSR setting the integral is evaluated without the singular point/pole, whereas in the MSR setting the singular integral could be evaluated also within the extra space that either houses the structural infinity (or just hosts the separated singular point when the unconventional division by zero is conceptually inapplicable there despite the quite unrealistic expectation that infinity is somehow inherited along with the complex numbers domain). For the complex analysis effectively shuns the infinity and routinely detours it. The traditional presentations of these and some related topics can be found in [49, 50].

Nevertheless, it is evident that the setvalued infinity could not be depicted directly in the very same primary space in which the singlevalued zero dwells and that is the chief reason for having multiple spaces, which is the fundamental tenet of the MSR framework.

4. COMPLEXIFIED OPERATION OF MULTIPLICATION BY ∞ THAT IS EQUATED WITH DIVISION BY 0

I have already proposed an algebraic multiplication of positive integer n by neverending ascending infinity as compound algebraic-differential operation that is equivalent to an algebraic division of the integer n by real zero, which, in general, should yield the expression

$$N_{n\uparrow}^{n\cdot\infty} \coloneqq n \cdot \infty = \frac{\mathrm{n}}{0} \eqqcolon N_{n\uparrow}^{n/0} \Leftrightarrow \sum_0^n \{[\int_1^\infty tv'([x],t)dt] \,|\, ȸ \oplus \left[\int_0^1 [\frac{1}{v'([x],t)dt}]\right] |ȹ\} \qquad (2)$$

which is definitely unrestricted operation, see [1]. The discrete multistage sum formula (2) is operationally similar to adding the primary integral that resembles formwise incomplete elliptic integral of the second kind to the paired inverted integral that resembles incomplete elliptic integral of the first kind, respectively, compare [2, 3]. Because of its apparent similarity to the formula (1), the far RHS of the formula (2) seemingly begs for rendering the whole formula (2) also in complex analytic form. The symbol ȸ stands for the homogeneous primary algebraic 3D basis and ȹ denotes the homogeneous 3D inverse/reciprocal basis.

Identifying the ascending infinity with natural inverse of zero, so that $n \cdot \infty = \frac{\mathrm{n}}{0}$ and the resulting from that identification indirect justification of the fact that $0 \cdot \infty = 1$ or even their selfevident divisions, such as $\frac{0}{0} = \frac{\infty}{\infty} = 1$ is not entirely new, but it was rather deliberately avoided and usually advised to take first their logarithms – see [51], as the SSR paradigm is unsuitable for any truly meaningful direct algebraic operations on infinity. Yet in the MSR setting one may operate on the real ascending infinity directly – see [3, 52, 53]. Recall that the clear equivalence $\frac{0}{0} = \frac{\infty}{\infty} = 1$ used as steppingstone for the conventional division by zero complies with the logically obvious intuition that dividing any entity by itself must yield unity.

The symbol ⊕ denotes composition implemented through an interspatial addition, which is different than the regular addition because the reciprocal term on the RHS of (2) cannot be quite meaningfully represented within the very same primary space [1, 3]. Because multiplication by ascending infinity designates a point at infinity with respect to the positive integer n and to the influence function v(x,t) – as it is sometimes called in the theory of integral kernels [54] – the operation (2) can be denoted in shorthand notation as $N_{n\uparrow}^{n\cdot\infty} \coloneqq n \cdot \infty = \frac{\mathrm{n}}{0} \eqqcolon N_{n\uparrow}^{n/0}$ by analogy to the point at infinity that is conventionally denoted by A_{∞}^{0} when it is used in reference to affine spaces [55] p.28. The up arrow that follows the integer n>0 signifies the fact that the integer variable n is progressing upwards n = 1,2,3…

The resulting two integrals on the RHS of (2) are formwise equivalent just as the preliminary conceptual evaluation of the intuitively selfevident informal symbolic implication

$$\infty+\infty=2\cdot\infty \Rightarrow 2/0 \tag{3}$$

seems to suggest. Their interspatial equivalence has been operationally confirmed in [2] after being conceptually introduced and already theoretically prepared in [3].

By analogy to operations on complex functions in conjunction with the Cauchy integral formula that is usually dubbed as CIF, one can see that the interspatial addition symbol $\oplus$ could also be used with the imaginary operator $\hat{\imath}$ because it virtually does split the formula (2) into as if multiple discrete stages of real and imaginary parts. Hence, I could rewrite (2) as

$$N_{n\uparrow}^{n\cdot\infty} \coloneqq n \cdot \infty = \frac{\mathrm{n}}{0} \eqqcolon N_{n\uparrow}^{\frac{n}{0}} \Longrightarrow \sum_{0}^{n}\{[\int_{1}^{\infty} tv'([x],t)dt] \,|\, \text{ȸ} \oplus \hat{\imath}\left[\int_{0}^{1} [\frac{1}{v'([x],t)dt}]\right] | \frac{\varphi}{\mathrm{i}} \} \tag{4}$$

in the complex analytic form, but as an implication to be justified elsewhere rather than as the equivalence appearing in (2). For at the present stage of the development of the topic the supposition (4) is not yet operationally warranted. Having said that, I shall respectfully mention that I am not really interested in ever proving an equivalence even if it would were possible, for the implication (4) reveals much more unanticipated features of feasibility of this unrestricted division by zero than the essentially algebraic operation alone entails.

The form of the compound nondenominated algebraic operator $\hat{\imath}|\frac{\varphi}{\mathrm{i}}$ is chosen on purpose for the sake of simplicity. For it preserves the inverse algebraic basis φ, so that the integral that stands by the – nondescript, at this stage of development of the present topic – algebraic operator $\hat{\imath}|\frac{\varphi}{\mathrm{i}}$ yields the integral's magnitude taken in the direction of the imaginary unit vector **i** that corresponds to the imaginary algebraic unit operator î as well as to the imaginary geometric unit operator $\vec{\imath}$, of course, so that in terms of values we have |î| =|$\vec{\imath}$| = |i| = √(-1). For if I would have chosen a denominated operator instead, as scalar product of the imaginary units, for instance, then our discussion here would require also some introduction to the spaces in which the operators operate on the functions, which is structural (i.e. the geometric or quasigeometric, not just operational/procedural) feat, that shall be further elaborated elsewhere. In this paper I wanted to focus only on purely algebraic aspects of the operational notion of infinity.

5. THE NEW SYNTHETIC METHODOLOGY FOR AVOIDING CONTROVERSIES BETWEEN OPERATIONAL AND STRUCTURAL REQUIREMENTS IN MATHEMATICS

Unrestricted algebraic operations, which should include unambiguous division by zero (if it is to be truly unrestricted), are needed for all four algebraic operations in order for them to be permissible and thus viable in realistic modeling of physical phenomena. This is because assigning numbers to points in a space is conventional and thus somewhat arbitrary. Hence the infamous – and unwarranted even though rarely contested – prohibition of division by zero is the main obstacle to any future development of physical sciences, no matter via which methods the division is going to be implemented in practice.

If unrestricted algebraic division by zero, which is necessary in the real spacetime, is implemented as algebraic multiplication by infinity in the domain of complex numbers, then zero and infinity should reside in separate mutually dual paired reciprocal spaces. One of the spaces is the primary unfurled three-dimensional (3D) algebraic Euclidean $\mathbb{R}^3$ space (commonly viewed as just a set-theoretical container of abstract number-points), whereas the other space of each pair is either a furled 1D or an unfurled 3D dual reciprocal space associated/paired with the primary space. Therefore, each of such mutually paired spaces should be equipped with its own (i.e. native to the given space) homogeneous orthogonal basis, whether algebraic or geometric. The pairing relies on reciprocity.

Since 3D homogeneous algebraic and geometric bases can preserve orthogonality intact the restriction of highest dimensionality of realistic simple geometric spaces to 4D does not preclude construction of higher-dimensional abstract quasispatial structures but it demands the deployment of heterogeneous 4D bases, for 3D homogeneous basis thwarts exceeding the 3^{rd} dimension without violating the symmetry that underlies the preservation of lengths (i.e. isometry). Nothing prevents construction of realistic higher dimensional spatial structures, but higher than the 3^{rd} dimension require partially overlaid structural composition such as the (3x+1t)D spacetime or the (1x+3t)D timespace that corresponds to, and partly overlays, the 4D spacetime; here x stands for length-based spatial variable(s) and t symbolizes time-based temporal variable(s). It may require thus pairing of single spaces that are called simple if they are equipped with an (algebraic or geometric) orthogonal 3D homogeneous basis.

However, realistic dimensions of constructible higher than 4D spatial structures should be quantized/grouped in triples (even if they are partly or entirely overlaid upon one another within a heterogeneous quadruple) if the structures are to remain credible. For dimensions are not like bricks that could be arbitrarily stacked upon each other at one's whim. They are abstract features of geometric and quasigeometric structures and as such they must be subject to both structural and operational rules satisfying the abstract structural and operational laws of mathematics, any violation of which is virtually punished by generating subtle yet often very confusing nonsenses, regardless of whether the nonsenses (or their consequences) are recognized as drivel or not.

Although the fact that contravariant representation of variables is reciprocal to that of covariant representation of the variables is well known and commonly accepted as valid in engineering applications involving vectors and tensors, and their handling of vectorial bases is correct [56], they do not always, if at all, keep track of the underlying them algebraic bases and of the need to change them when it becomes necessary. For if we substitute for covariant derivatives their reciprocals, we obtain a set of equations connecting the elements of the [inverse/reciprocal] contravariant system [57]. Reciprocity is fundamental even though its importance was ignored. Nevertheless, thinking in terms of tensor representations (also known as "absolute differential calculus" in the past) may inadvertently lead to incorrect inferences in other than purely radial cases.

In this paper I have focused on the new operational treatment of the notion of infinity and was concentrated on complex analysis as a realistic (not to say "real") way to deduce properties of the operational infinity. For quick review of main results of complex analysis for physical sciences see [58]. Even Euler, who invented numerous formulas involving real, imaginary and complex numbers, once apparently also considered imaginaries as impossible quantities [59]. Yet they are not only possible but indispensable for understanding the notions of spatiality in general, and dimensionality in particular. My (new) synthetic approach to mathematics builds (or synthesizes) theories not upon a competitive fight of contesting ideas but on coexistence of structurally valid concepts (if the structures comply with the operations that are supposed to fit the structures) has guided this presentation. It is because matching operations to the structures on which one wants to operate, is the essence of the new synthetic approach that I have proposed in many of my papers. For there is certainly neither need nor advantage to operate on impossible to construct, unrealistic though postulated structures or for trying to build fictitious quasigeometric structures if one could not legitimately operate on them. The importance of matching constructible structures with the operational procedures that should correspond to the structures is intuitively clear, I think.

Yet the new synthetic approach that I have espoused was co-inspired by the MacLane's tradition-breaking assertion that "the fundamental object of mathematics is not a set composed of elements, but a sheaf of functions on some nonspecified space or locale" [60], which – to me – has underscored the primary importance of local variations of the sheaves of functions and consequently thus also the necessity to treat them in most abstract terms of operationally legitimate differentials. My personal impression was that he called for such mathematical models of spaces whose validity is to be ensured by the feasibility of operations performed on the functions' sheaves. Recall that the traditional approach relied on three fundamental processes commonly found in mathematics, namely: constructing objects, then forming relations between the objects, and finally demonstrating that certain such relations are true (i.e. proving theorems) – compare [61]. But there was neither the assurance that the postulated objects were indeed constructible (not merely declared as being unambiguously constructible) nor any guarantee that the objects can be meaningfully operated on. Everything regarding the correctness had implicitly relied on the attained skills and the current knowledge of the mathematician/constructor.

Yet even without doubting the ingenuity or expertise of the mathematician, there was no provision for shifting paradigms in the traditional mathematics. Since formerly unanticipated experimental results virtually compel us to shift some previously unchallenged paradigms, we should devise an internal mathematical metaprocess for validating all the necessary modifications without reinventing everything from scratch under the new set of paradigms. And that is the chief role of the new synthetic approach to mathematical and mathematized sciences.

The synthetic approach permits the following interpretation of the formula (4).

Since gradient represents derivative and the imaginary unit operator represents twist then the formula (4) is an integral analogue of the geometric differential operator GDiff F()

$$GDiff\ F(P) = AGrad\ F(P) + \ GTwst(F(P), \dots) \tag{5}$$

which has been discussed in [62], see formulas (28, 29) and those following them. The AGrad is an algebraic gradient operator and GTwst is the geometric twist operator acting on the given function F(P) of the given primary space P. The eq. (5) just alludes to the supposition that the formula (4) can make sense also in the framework of extended differential operators. Since the domain of complex numbers is only 2D algebraic structure, further discussion of the topic shall be presented elsewhere, preferably deferred to the context of 4D quaternions.

6. CONCLUSIONS

I have shown by examples that the notion of an operational infinity can also be interpreted as a complex algebraic entity combining the real neverending ascending infinity with its imaginary infinitesimal reciprocal counterpart that corresponds to the descending infinity. Hence the twin character of fully operational infinity is depicting not only a dual reciprocal procedure but possibly also prospective structure of multispatial nature of the essentially setvalued infinity, which shall be discussed elsewhere. Hence the fully operational/procedural infinity is a compound entity of complex character and clearly twin dual reciprocal nature.

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