Available online at www.worldscientificnews.com

World Scientific News

An International Scientific Journal

WSN 145 (2020) 274-285 EISSN 2392-2192

Two Modulo Three Sum Graphs

M. Vanu Esakki[1] and **M. P. Syed Ali Nisaya**[2]

P.G. & Research Department of Mathematics, The M. D. T. Hindu College
Tirunelveli - 627010, Tamil Nadu, India

[1,2]E-mail address: vanuesakki@gmail.com , nisayarifoy@gmail.com

ABSTRACT

Let G = (V, E) be a graph with p vertices and q edges is said to be a two modulo three sum graph if there is an injective function f from V(G) to $\{ a : 0 \leq a \leq 3q-1$ and either $a \equiv 0 (mod\ 3)\ or\ a \equiv 2 (mod\ 3)\}$ where q is the number of edges of G and such that f induces a bijection f^* from E(G) to $\{a : 2 \leq a \leq 3q-1\ and\ a \equiv 2(mod\ 3)\}$ given by $f^*(uv) = f(u) + f(v)$ and the function f is called two modulo three sum labeling of G. In this paper, we introduce an analog of sum labeling known as two modulo three sum labeling and we define two modulo three sum labeling of some tree related graphs. Also we prove that split star, mirror path graph, complete bipartite graph and $C_4 \odot nK_1$ are two modulo three sum graphs.

Keywords: Two modulo three sum labeling, Two modulo three sum graph

1. INTRODUCTION AND DEFINITIONS

The graph considered in this paper are finite, undirected and without loops or multiple edges. Let G = (V, E) be a graph with p vertices and q edges. Terms not defined here are used in the sense of Harary [10], Parthasarathy [19] and Bondy and U.S.R. Murthy [4]. For number theoretic terminology, we refer to [2] and [18].

Graph labeling is one of the fascinating areas of graph theory with wide ranging applications. Graph labeling was first introduced in 1960's. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the

(Received 20 April 2020; Accepted 08 May 2020; Date of Publication 09 May 2020)

mapping is the set of vertices (edges / both) then the labeling is called the vertex (edge / total) labeling. Most popular graph labeling trace their origin to one introduced by Rosa [20]. Rosa called a function (labeling) f a β-valuation of a graph in the year 1966 and Golomb [9] called it as graceful labeling. There are several types of graph labeling and a detailed survey is found in [6].

The concept of a sum graph was introduced by Harary [11] in 1990 and was defined as a graph whose vertices can be labeled with distinct positive integers so that the sum of the labels on each pair of adjacent vertices is the label of some other vertex. In 1991, Harary et al. [12] defined a real sum graph. One of the earliest interesting results was due to Ellingham [5] who proved the conjecture of Harary [11]. For more information about sum graphs, see [13].

In [3], the concept of odd sum labeling was introduced. Jeyanthi et al. [14] introduced one molulo three meanlabeling. For more information related to one modulo three mean labeling and one modulo N mean labeling, see [7], [8], [15], [16] and [17]. The following definitions are necessary for present study.

Definition: 1.1. A Path $P_n$is obtained by joining u_i to the consecutive vertices u_{i+1} for $1 \leq i \leq n-1$.

Definition: 1.2. The graph mG is m copies of the graph G.

Definition: 1.3. The Corona of two graphs G_1 and G_2 is the graph $G = G_1 \odot G_2$ formed by one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th}vertex of G_1 adjacent to every vertex in thei^{th} copy of G_2.

Definition: 1.4. Let P_n be the path on n vertices. Then the Twig graph obtained from the path P_n by attaching exactly two pendent edges to each internal vertex of the path and it is denoted by $TW(P_n)$.

Definition: 1.5. The Coconut tree CT(m, n) is a graph obtained from the path P_m by appending n new pendent edges at an end vertex of P_m.

Definition: 1.6. The Bistar $B(m,n)$ is a graph obtained from P_2 by joining m pendent edges to one end of P_2 and n pendent edges to the other end of P_2. The edge of P_2 is called the central edge of $B(m,n)$ and the vertices of P_2 are the called the central vertices of $B(m,n)$.

Definition: 1.7 The Subdivision of an edge e = uv of a graph G is the replacement of the edge e by a path (u, v, w). If every edge G is subdivided exactly once, then the resulting graph is called a subdivision graph S(G).

Definition:1.8 The **Bi-graph** (bipartite graph) G is a graph whose vertex set V can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 and a vertex of V_2.

Definition: 1.9 The Complete bipartite graph is a bipartite graph with bipartition (V_1, V_2) such that every vertex of V_1 is joined to all the vertices of V_2. It is denoted by $K_{m,n}$ where $|V_1| = m$ and $|V_2| = n$.

Definition: 1.10. The complete bipartite graph $K_{1,n}$ is called a Star graph.

Definition: 1.11. The Y – Tree is a graph obtained from path by appending an edge to a vertex of a path adjacent to an end point and it is denoted Y_n where n is the number of vertices in the tree.

Definition: 1.12.[1]. The H-graph of a path P_n is a graph obtained from two copies of P_n with vertices $u_1, u_2, \ldots, u_n$ and $v_1, v_2, \ldots, v_n$ by joining the vertices $u_{(n+1)/2}v_{(n+1)/2}$ if n is odd and $u_{\frac{n}{2}+1}v_{n/2}$ if n is even. It is denoted by $H(P_n)$.

Definition: 1.13. For a graph G, its Split graph is obtained by adding to each vertex v, a new vertex v' so that v' is a adjacent to every vertex that is adjacent to v in G. It is denoted by $S(G)$.

Definition: 1.14. Let G be a graph. Let G' be a copy of G. The Mirror graph M(G) of G is defined as the disjoint union of G and G' with additional edges joining each vertex of G to its corresponding vertex in G'.

Definition: 1.15. A closed trail whose origin and interval vertices are distinct is called a C**ycle.** A cycle of length n is called n-cycle. It is denoted by C_n.

2. MAIN RESULTS

Definition: 2.1. Let G = (V, E) be a graph with p vertices and q edges is said to be a two modulo three sum graph if there is an injective function f from V(G) to $\{a : 0 \leq a \leq 3q - 1 \text{ and either } a \equiv 0(mod\ 3) \text{ or } a \equiv 2(mod\ 3)\}$where q is the number of edges of G and such that $f$induces a bijectionf^* from E(G) to $\{a : 2 \leq a \leq 3q - 1 \text{ and } a \equiv 2(mod\ 3)\}$ given by$f^*(uv) = f(u) + f(v)$ and the function f is called two modulo three sum labeling of G.

Example 2.2 Two modulo three sum labeling of C_4 is given in figure 1.

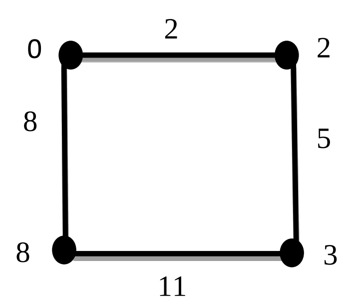

Figure 1.

Theorem: 2.3. Any path P_n is a two modulo three sum graph.

Proof: Let G be a path P_n.

Let $V(G) = \{v_i : 1 \leq i \leq n\}$ and $E(G) = \{e_i = v_i v_{i+1} : 1 \leq i \leq n-1\}$.

Here G has n vertices and $n-1$ edges. Define $f : V(G) \rightarrow \{0, 2, 3, \ldots, 3n-4\}$ as follows.

$$f(v_i) = \begin{cases} \frac{3(i-1)}{2} & if\ i\ is\ odd\ and\ 1 \leq i \leq n \\ \frac{3i-2}{2} & if\ i\ is\ even\ and\ 2 \leq i \leq n \end{cases}$$

Clearly f is injective and f induces a bijective function $f^* : E(G) \rightarrow \{2, 5, 8, \ldots, 3n-4\}$ as $f^*(e_i) = 3i-1, 1 \leq i \leq n-1$.Hence the edge labels are $2, 5, 8, \ldots, 3n-4$.Thus f is a two modulo three sum labeling of G. Therefore G = P_n is a two modulo three sum graph.

Theorem: 2.4. Any star $K_{1,n}$ is a two modulo three sum graph.

Proof: Let G be a star $K_{1,n}$.

Let $V(G) = \{v, v_i : 1 \leq i \leq n\}$ and $E(G) = \{e_i = vv_i : 1 \leq i \leq n\}$.

Here G has $n+1$ vertices and n egdes.Define $f : V(G) \rightarrow \{0, 2, 3, \ldots, 3n-1\}$ as follows.

$f(v) = 0$ and $f(v_i) = 3i-1, 1 \leq i \leq n$

Clearly f is injective and $f$induces a bijective function$f^* : E(G) \rightarrow \{2, 5, \ldots, 3n-1\}$ as follows.

$f^*(e_i) = 3i-1, 1 \leq i \leq n$. Hence the edge labels are $2, 5, \ldots, 3n-1$. Therefore f is a two modulo three sum labeling. Thus $G = K_{1,n}$ is a two modulo three sum graph.

Theorem: 2.5. Any subdivision graph $S(K_{1,n})$ is a two modulo three sum graph.

Proof: Let G be a subdivision graph.

Let $V(G) = \{v, v_i, u_i : 1 \leq i \leq n\}$ and $E(G) = \{vv_i, v_i u_i : 1 \leq i \leq n\}$.

Here 2n+1 vertices and 2n edges.

Define $f : V(G) \rightarrow \{0, 2, \ldots, 6n-1\}$ as follows.

$f(v) = 0$

$f(v_i) = 3i-1, \ 1 \leq i \leq n$

$f(u_i) = 6n-6i+3, \ 1 \leq i \leq n$

Clearly f is injective and f induces a bijective function $f^*: E(G) \rightarrow \{2, 5, \ldots., 6n-1\}$ as follows.

$f^*(vv_i) = 3i-1, \ 1 \leq i \leq n$

$f^*(v_i u_i) = 6n-3i+2, \ 1 \leq i \leq n$

Hence the edge labels are $2, 5, \ldots., 6n-1$. Thus f is a two modulo three sum labeling of G.

Therefore G $= S(K_{1,n})$ is a two modulo three sum graph.

Theorem: 2.6. Any $P_n \odot mK_1$ is a two modulo three sum graph where $m \geq 2$ and $n \geq 2$.

Proof: Let G be a $P_n \odot mK_1$ graph where $m \geq 2$ and $n \geq 2$.

Let $V(G) = \{v_i, v_{ij} : 1 \leq i \leq n \text{ and } 1 \leq j \leq m\}$ and

$E(G) = \{v_i v_{i+1}, v_i v_{ij} : 1 \leq i \leq n \,\&\, 1 \leq j \leq m\}$.

Here G has mn vertices and mn + n- 1 edges.

Define $V(G) \rightarrow \{0,2, \ldots, 3mn + 3n - 4\}$ as follows.

$$f(v_i) = \begin{cases} \dfrac{3(i-1)(m+1)}{2} & \text{if } i \text{ is odd and } 1 \leq i \leq n \\ \dfrac{3i(m+1)-2}{2} & \text{if } i \text{ is even and } 2 \leq i \leq n \end{cases}$$

For $1 \leq i \leq n, f(v_{ij}) = \begin{cases} \frac{3i(m+1)-6(m+1-j)}{2} & \text{if } i \text{ is even and } j = 1,2, \ldots, m \\ \frac{3i(m+1)-3m+6j-1}{2} & \text{if } i \text{ is odd and } j = 1,2, \ldots, m \end{cases}$

Clearly f is injective and f induces a bijective function $f^* : E(G) \rightarrow \{2,5, \ldots, 3mn + 3n - 4\}$ as

For $1 \leq i \leq n,\ f^*(v_i v_{ij}) = \begin{cases} \frac{6i(m+1)-6(m-j)-8}{2} & \text{if } i \text{ is even and } j = 1,2, \ldots, m \\ 3(i-1)(m+1) + 3j - 1 & \text{if } i \text{ is odd and } j = 1,2, \ldots, m \end{cases}$

For $1 \leq i \leq n-1,\ f^*(v_i v_{i+1}) = 3mi + 3i - 1$

Thus f is a two modulo three sum labeling of G. Therefore G= $P_n \odot mK_1$ is a two modulo three sum graph where $m \geq 2$ and $n \geq 2$.

Theorem: 2.7. Any Twig graph $TW(P_n)$ is a two modulo three sum graph.

Proof: Let G be a Twig graph $TW(P_n)$.

Let $V(G) = \{v_i, u_j, w_j : 1 \leq i \leq n \text{ and } 2 \leq j \leq n-1\}$ and

$E(G) = \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_j u_j, v_j w_j : 2 \leq i, j \leq n-1\}$.

Here G has $3n - 4$ vertices and $3n - 5$ edges.

Define $f : V(G) \rightarrow \{0,2,3, \ldots, 9n - 16\}$ as follows.

For $1 \leq i \leq n, f(v_i) = \begin{cases} \frac{9i-9}{2} & \text{if } i \text{ is odd} \\ \frac{9i-14}{2} & \text{if } i \text{ is even} \end{cases}$

$f(u_j) = \begin{cases} 6j - 9 & \text{if } j \text{ is even and } 2 \leq j \leq n-1 \\ 6j - 13 & \text{if } j \text{ is odd and } 3 \leq j \leq n-1 \end{cases}$ and

$f(w_j) = \begin{cases} 3j - 1 & \text{if } j \text{ is odd } 3 \leq j \leq n-1 \\ 3j & \text{if } j \text{ is even } 2 \leq j \leq n-1 \end{cases}$

Clearly f is injective and f induces a bijective function $f^*: E(G) \to \{2, 5, \dots, 9n-16\}$ as $f^*(v_i v_{i+1}) = 9i - 7, \quad 1 \le i \le n$; $f^*(v_j u_j) = 9j - 4, 2 \le j \le n-1$and $f^*(v_j w_j) = 9j - 1$, $2 \le j \le n-1$. Hence the edge labels are $2, 5, \dots, 9n-16$. Thus f is a two modulo three sum labeling of G.Therefore G = $TW(P_n)$ is a two modulo three sum graph.

Theorem: 2.8. Any coconut tree CT(m, n) is a two modulo three sum graph.

Proof: Let G be a coconut tree graph CT(m, n) .

Let $V(G) = \{v_i, u_j : 1 \le i \le m \;\&\; 1 \le j \le n\}$ and

$E(G) = \{v_i v_{i+1} : 1 \le i \le m-1\} \cup \{v_1 u_j : 1 \le j \le n\}$.

Here G has $m+n$ vertices and $m+n-1$edges.

Define $f: V(G) \to \{0, 2, \dots, 3m+3n-4\}$ as follows.

$f(u_j) = 3j - 1, \;\; 1 \le j \le n$ and

For $1 \le i \le m$, $f(v_i) = \begin{cases} \frac{3i+2(3n-1)}{2} & if\ i\ is\ even \\ 3(i-1) & if\ i\ is\ odd \end{cases}$

Clearly f is injective and f induces a bijective function $f^*: E(G) \to \{2, 3, \dots, 6n-4\}$ as follows.

$f^*(v_1 u_j) = 3j - 1, \quad 1 \le j \le n$ and $f^*(v_i v_{i+1}) = 3n + 3i - 1, 1 \le i \le m-1$

Hence the edge labels are $2, 3, \dots, 6n-4$. Thus f is a two modulo three sum labeling of G.

Therefore coconut tree is a two modulo three sum graph.

Theorem: 2.9. Any Y – tree is a two modulo three sum graph.

Proof: Let G be a Y – Tree.

Let $V(G) = \{v_i : 1 \le i \le n\}$and $E(G) = \{v_i v_{i+1} : 1 \le i \le n-2 \ and\ v_{n-2} v_n\}$.

Here G has n vertices and n – 1 edges.

Define $f: V(G) \to \{0, 2, \dots 3n-4\}$ as follows.

$f(v_i) = \begin{cases} \frac{3(i-1)}{2} & if\ i\ is\ odd\ and\ 1 \le i \le n-1 \\ \frac{3i-2}{2} & if\ i\ is\ even\ and\ 1 \le i \le n-1 \end{cases}$ and

$f(v_n) = f(v_{n-1}) + 3$

Clearly f is injective and f induces a bijective function $f^* : E(G) \to \{2, 5, 8, \dots, 3n-4\}$ as $f^*(e_i) = 3i - 1, 1 \le i \le n-2$ and $f^*(e_{n-1}) = f(v_{n-2}) + f(v_n)$.

Hence the edge labels are $2, 5, 8, \dots, 3n-4$. Thus f is a two modulo three sum labeling of G.

Therefore G = Y - Tree is a two modulo three sum graph.

Theorem: 2.10. Any $H(P_n)$ graph is a two modulo three sum graph.

Proof: Let G be a $H(P_n)$graph. Let$V(G) = \{v_i, u_i : 1 \leq i \leq n\}$ and $E(G) = \{v_i v_{i+1}, u_i u_{i+1} : 1 \leq i \leq -1\} \cup \left\{u_{\frac{n+1}{2}} v_{\frac{n+1}{2}} \ if \ \ n \ is \ odd \ \ \& \ u_{\frac{n}{2}+1} \ \ v_{\frac{n}{2}} \ if \ \ n \ is \ even\right\}$. Here 2n vertices and 2n-1 edges. Define $f: V(G) \to \{0,2,3, \dots, 6n-4\}$ as follows.

Case (i). n is odd

For $1 \leq i \leq n,$ $\quad f(v_i) = \begin{cases} \frac{3(i-1)}{2} & if \ i \ is \ odd \\ \frac{3i-2}{2} & if \ i \ is \ even \end{cases}$

For $1 \leq i \leq n,$ $\quad f(u_i) = \begin{cases} \frac{3n+3i-2}{2} & if \ i \ is \ odd \\ \frac{3n+3i-3}{2} & if \ i \ is \ even \end{cases}$

Case (ii). n is even

For $1 \leq i \leq n,$ $\quad f(v_i) = \begin{cases} \frac{3(i-1)}{2} & if \ i \ is \ odd \\ \frac{3i-2}{2} & if \ i \ is \ even \end{cases}$

For $1 \leq i \leq n,$ $\quad f(u_i) = \begin{cases} \frac{3n+3i-3}{2} & if \ i \ is \ odd \\ \frac{3n+3i-2}{2} & if \ i \ is \ even \end{cases}$

Clearly f is injective and f induces a bijective function $f^*: E(G) \to \{2,5, \dots, 6n-4\}$.

Hence the edge labels are $2,5, \dots, 6n-4$. Thus f is a two modulo three sum labeling of G.

Therefore $H(P_n)$ grpah is a two modulo three sum graph.

Theorem: 2.11. Any $H(P_n) \odot mK_1$ is a two modulo three sum graph.

Proof: Let G be a $H(P_n) \odot mK_1$ graph.

Let $V(G) = \{v_i, u_i : 1 \leq i \leq n\} \cup \{v_{ij}, u_{ij} : 1 \leq i \leq n \ \ and \ 1 \leq j \leq m\}$ and

$E(G) = \{v_i v_{i+1}, u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{ij}, u_i u_{ij} : 1 \leq i \leq n \ and \ 1 \leq j \leq m\}$.

Here G has 2n(m+1) vertices and 2n-1+2mn edges. Define $V(G) \to \{0,2, \dots, 6n(m+1) - 4\}$ as follows.

For $1 \leq i \leq n,$ $\quad f(v_i) = \begin{cases} \frac{3(i-1)(m+1)}{2} & if \ i \ is \ odd \\ \frac{3i(m+1)-2}{2} & if \ i \ is \ even \end{cases}$

For $1 \leq i \leq n \ \ and$ For $1 \leq j \leq m,$

$$f(v_{ij}) = \begin{cases} \dfrac{3i(m+1) - 6(m+1-j)}{2} & if \ i \ is \ even \\ \dfrac{3(i-1)(m+1) + 6j - 2}{2} & if \ i \ is \ odd \end{cases}$$

Case (i). n is odd

For $1 \le i \le n$, $f(u_i) = \begin{cases} \frac{(m+1)(3n+3i)-2}{2} & if\ i\ is\ odd \\ \frac{(m+1)(3n+3(i-1))}{2} & if\ is\ even \end{cases}$

For $1 \le i \le n$ and For $1 \le j \le m$,

$$f(u_{ij}) = \begin{cases} \frac{(m+1)(3n+3i)-6m+6(j-1)}{2} & if\ is\ odd \\ \frac{(m+1)(3n+3(i-1))+4+6(j-1)}{2} & if\ i\ is\ even \end{cases}$$

Case (ii). n is even

For $1 \le i \le n$, $f(u_i) = \begin{cases} \frac{3n(m+1)+(3m+3)(i-1)}{2} & if\ i\ is\ odd \\ \frac{3n(m+1)-2+(3m+3)i}{2} & if\ i\ is\ even \end{cases}$

For $1 \le i \le n$ and For $1 \le j \le m$,

$$f(u_{ij}) = \begin{cases} \frac{3n(m+1)+(3m+3)(i-1)+4+6(j-1)}{2} & if\ i\ is\ odd \\ \frac{3n(m+1)+(3m+3)i-6m+6(j-1)}{2} & if\ i\ is\ even \end{cases}$$

Clearly f is injective and f induces a bijective function $f^*: E(G) \to \{2,5, \dots ,6n(m+1)-4\}$. Thus f is a two modulo three sum labeling of G. Therefore G = $H(P_n) \odot mK_1$ is a two modulo three sum graph.

Theorem: 2.12. Any bistar graph $B(m,n)$ is a two modulo three sum graph.

Proof: Let G be a bistar graph $B(m,n)$. Let $V(G) = \{v, v_i, u, u_j : 1 \le i \le m\ and\ 1 \le j \le n\}$ and $E(G) = \{uv,\ vv_i : 1 \le i \le m\ \ and\ uu_j : 1 \le j \le n\}$.

Here G has $m+n+2$ vertices and

$m+n+1$ edges. Define $f : V(G) \to \{0, 2, 3, \dots,\ 3m+3n+2\}$ as follows.

$f(v) = 0$

$f(v_i) = 3i - 1, \quad 1 \le i \le m$

$f(u) = 3m + 2$

$f(u_j) = 3j, \quad 1 \le j \le n$

Clearly f is injective f induces a bijective function $f^* : E(G) \to \{2, 5, \dots,\ 3m+3n+2\}$ as follows. $f^*(vv_i) = 3i - 1, 1 \le i \le m$

$f^*(uv) = 3m + 2$

$f^*(uu_j) = 3m + 3j + 2, 1 \le j \le n$

Hence the edge labels are $2, 5, \ldots, 3m + 3n + 2$. Thus f is a two modulo three sum labeling. Therefore $B(m, n)$ is a two modulo three sum graph.

Theorem: 2.13. Any split star graph $S(K_{1,n})$ is a two modulo three sum graph.

Proof: Let G be a split star graph $S(K_{1,n})$.

Let $V(G) = \{v, v', v_i, v_i' : 1 \le i \le n\}$ and $E(G) = \{vv_i, vv_i', v'v_i : 1 \le i \le n\}$.

Here G has 2n+2 vertices and $3n$ edges.

Define $f : V(G) \to \{0, 2, 3, \ldots, 9n - 1\}$ as follows.

$f(v) = 0$

$f(v') = 6n$

$f(v_i) = 3i - 1, \quad 1 \le i \le n$

$f(v_i^i) = 3n + 3i - 1, \quad 1 \le i \le n$

Clearly f is injective and f induces a bijective function $f^* : E(G) \to \{2, 5, \ldots, 9n - 1\}$ as follows.

$f^*(vv_i) = 3i - 1, 1 \le i \le n.$

$f^*(vv_i') = 3n + 3i - 1, 1 \le i \le n$

$f^*(v'v_i) = 6n + 3i - 1, 1 \le i \le n$

Hence the edge labels are $2, 5, \ldots, 9n - 1$. Thus f is a two modulo three sum labeling.

Therefore $G = S(K_{1,n})$ is a two modulo three sum graph.

Theorem: 2.14. Any mirror path is a two modulo three sum graph.

Proof: Let G be a mirror path $M(P_n)$ graph.

Let $V(G) = \{v_i, v_i' : 1 \le i \le n\}$ and $E(G) = \{v_i v_{i+1}, v_i' v_{i+1}' : 1 \le i \le n - 1\} \cup \{v_i v_i' : 1 \le i \le n\}$. Here 2n vertices and 3n - 2 edges. Define $f : V(G) \to \{0, 2, 3, \ldots, 9n - 7\}$ as follows.

For $1 \le i \le n$, $f(v_i) = \begin{cases} \frac{3(i-1)}{2} & if\ i\ is\ odd \\ \frac{3i-2}{2} & if\ i\ is\ even \end{cases}$

For $1 \le i \le n$, $f(v_i') = \begin{cases} \frac{6n+3i-5}{2} & if\ i\ is\ odd \\ \frac{6n+3(i-2)}{2} & if\ i\ is\ even \end{cases}$

Clearly f is injective and f induces a bijective function $f^*: E(G) \to \{2, 5, \ldots, 9n - 7\}$ as follows.

$f^*(v_i v_{i+1}) = 3i - 1, \ 1 \le i \le n - 1$

$$f^*(v_i'v_{i+1}') = \frac{12n+6i-8}{2}, \quad 1 \le i \le n-1$$

$$f^*(v_iv_i') = \frac{6n+6i-8}{2}, \quad 1 \le i \le n$$

Hence the edge labels are $2,5,\dots,9n-7$. Thus f is a two modulo three sum labeling of G.

Therefore $G = M(P_n)$ is a two modulo three sum graph.

Theorem: 2.15. Any complete bipartite graph $K_{n,m}$ is a two modulo three sum graph.

Proof: Let G be complete bipartite graph $K_{n,m}$.

Let $V(G) = \{v_i, u_j : 1 \le i \le n \ and \ 1 \le j \le m\}$ and

$E(G) = \{ v_1u_j, v_2u_j, \dots, v_nu_j: 1 \le j \le m\}$.

Here G has $n+m$ vertices and mn edges.

Define $f : V(G) \to \{0, 2, 3, \dots, 3mn-1\}$ as follows.

$f(v_i) = 3m(i-1), \ 1 \le i \le n$

$f(u_j) = 3j-1, \quad 1 \le j \le m$

Clearly f is injective and f induces a bijective function $f^* : E(G) \to \{2, 5, 8, \dots, 3mn-1\}$.

Hence the edge labels are $2, 5, 8, \dots, 3mn-1$. Thus f is a two modulo three sum labeling of G.

Therefore G = $K_{n,m}$ is a two modulo three sum graph.

Theorem: 2.16. Any $C_4 \odot nK_1$ is a two modulo three sum graph.

Proof: Let G be a $C_4 \odot nK_1$ graph.

Let $V(G) = \{v_i, v_{ij}: 1 \le i \le 4 \ and \ 1 \le j \le n\}$ and

$E(G) = \{v_iv_{i+1}: 1 \le i \le 3\} \cup \{v_1v_4\} \cup \{v_iv_{ij}: 1 \le i \le 4 \ and \ 1 \le j \le n\}$.

Here G has 4n+4 vertices and 4n+4 edges.

Define $V(G) \to \{0,2,\dots,12n+11\}$ as follows.

$f(v_1) = 0$

$f(v_2) = 2$

$f(v_3) = 3$

$f(v_4) = 8$

$f(v_{1j}) = 3n + 3(j-1) + 2, \quad 1 \le j \le n$

$f(v_{2j}) = 3n + 3j + 9, \quad 1 \le j \le n$

$f(v_{3j}) = 6n + 3j + 8, \ 1 \le j \le n$

$f(v_{4j}) = 9n + 3j + 3, \quad 1 \leq j \leq n$

Clearly f is injective and f induces a bijective function $f^*: E(G) \rightarrow \{2,5,\ldots,12n+11\}$. Hence the edge labels are $2, 5, 8, \ldots, 12n + 11$. Thus f is a two modulo three sum labeling of G.

Therefore $G = C_4 \odot nK_1$ is a two modulo three sum graph.

Observation: 2.17. In every two modulo three sum graph G, the vertices with label 0 and 2 are always adjacent.

Proof: The edge label 2 is possible only when the vertices with label 0 and 2 are always adjacent.

3. CONCLUSION

In this paper, we have studied the two modulo three sum labeling of some tree related graphs. This work contributes several new results to the theory of graph labeling.

ACKNOWLEDGEMENTS

Authors are thankful to the anonymous reviewer for the valuable comments and suggestions that improve the quality of this paper.

References

[1] B. Amutha and M.P. Syed Ali Nisaya, Hexagonal Sum Labeling of H-graphs. *Adalya Journal*, 8(10), 2019, 33-39

[2] M. Apostal, Introduction to Analytic Number Theory, Narosa Publishing House, Second Edition, 1991

[3] S. Arockiaraj and P. Mahalakshmi, On Odd Sum Graphs, *International Journal of Mathematical Combinatorics*, 4 (2013) 58-77

[4] J. A. Bondy and U. S. R. Murty. Graph Theory with Applications, Macmillan Press, London (1976).

[5] M. Ellingham, Sum Graphs from Trees. *Ars Combinatoria*, 35 (1993) 335-349

[6] J.A. Gallian, A Dynamic Survey of Graph Labeling. *The Electronic Journal of Combinatorics*, 22 (2019) #DS6

[7] B. Gayathri and V. Prakash, Necessary conditions for one modulo three mean labeling of graphs. *International Journal of Engineering Sciences, Advanced Computing and Bio-Technology*, 6(2) (2015) 41-61

[8] B. Gayathri and V. Prakash, One modulo three mean labeling of some special graphs. *Mathematical Sciences International Research Journal*, 4(2), 2015, 434-460

[9] S.W. Golomb, How to number a Graph, Graph Theory and Computing, R.C. Read, Academic Press, New York (1972)

[10] F. Harary, Graph Theory, Adision – Wesley, Reading Mass, 1969

[11] F. Harary, Sum Graphs and Difference Graphs. *Congr. Numer.* 72 (1990) 101-108

[12] F. Harary, I. R. Hentzel and D. Jacobs Digitizing Sum Graphs over the Reals, *Caribbean J. Math. Comput. Sci.* 1 (1991) 1-4

[13] F. Harary, Sum Graphs over all integers. *Discrete Mathematics* 124 (1994) 99-105

[14] P. Jeyanthi and A. Maheswari, One modulo three mean labeling of graphs. *American Journal of Applied Mathematics and Statistics*, 2(5) (2014) 302-306.

[15] P. Jeyanthi, A. Maheswari and P. Pandiraj, One Modulo Three Mean Labeling of Cycle Related graphs. *International Journal of Pure and Applied Mathematics*, Vol. 103, No. 4, 2015, 625-633

[16] P. Jeyanthi, A. Maheswari and P. Pandiraj, On one modulo three mean labeling of graphs. *Journal of Discrete Mathematical sciences and Cryptography*, 19(2) (2016) 375 –384

[17] D. Muthuramakrishanan and K. Bhagyasri, One modulo N mean Labeling of Graphs. *International Journal of Scientific Research and Review*, 7(8), 2018, 345-354

[18] I. Niven and Herbert S. Zuckerman, An Introduction to the Theory of Numbers, Wiley Eastern Limited, Third Edition, 1991

[19] K. R. Parthasarathy. Basic Graph Theory, Tata Mcgraw Hill Publishing Company Limited (1994).

[20] A. Rosa, On certain valuation of the vertices of a graph, Thoery of Graphs (Proc. Internat. Symposium, Rome, 1966), Gordon and Breach, N. Y and Dunod Paris (1967), 349-355