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Some Algebraic Structures of Picture Fuzzy Matrices

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ABSTRACT

In this paper, the concept of Picture fuzzy matrices, which are direct extensions of an intuitionistic fuzzy matrices. Then we define some algebraic operations of Picture fuzzy matrices, such as max-min, min-max, complement, algebraic sum, algebraic product, scalar multiplication (nA) and exponentiation (A^n). We also investigate their algebraic properties of these operations. Furthermore, we define a new operation(@) on Picture fuzzy matrices and discuss distributive laws in the case where the operations of $\oplus_{\wp}$, $\otimes_{\wp}$, $\wedge_{\wp}$ and $\vee_{\wp}$ are combined each other.

***Keywords*:** Intuitionistic fuzzy matrix, Picture fuzzy matrix, Algebraic sum, Algebraic product, Scalar multiplication, Exponentiation operations

1. INTRODUCTION

The concept of the intuitionistic fuzzy matrix (IFM) theory [5] is the extension of the fuzzy matrix theory [13]. Since the appearance of IFM, several researchers have importantly contributed to the development of IFM theory and its applications [3, 4, 6-8, 10]. In intuitionistic fuzzy matrices, only membership and non-membership are considered. In different branches of social sciences, sciences and medical sciences, it was found that two components are not sufficient to represent some special types of information. In such cases, a component, namely neutrality, is needed to represent the information completely. As, for example, in medical science, a disease may have three types of effects (positive, negative and

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neutral) on a particular symptom, i.e. to measure how much a disease positively, negatively and neutrally effects a particular symptom, three components are required. Thus, to remove the limitation of intuitionistic fuzzy matrix and to handle more possible types of uncertainty in practical situation, picture fuzzy matrix was initiated by Dogra and Pal [2] as a generalization of intuitionistic fuzzy matrix. Therefore, the focus of this paper, we define Algebraic operations of Picture fuzzy matrices (PFM) and develop some basic algebraic properties of PFM and investigated their desirable properties.

The part of this paper is as follows. In *Algebraic properties on PFMs* section, we define Algebraic operations of Picture fuzzy matrices and its basic properties are proved. In *New operation(@) on Picture fuzzy matrices* section, we define a new operation(@) on Picture fuzzy matrices and discuss some their properties. we write the *Conclusion* of the paper in the last section.

Definition 1.1. [5] An intuitionistic fuzzy matrix (IFM) is a matrix of pairs $A = \left(\langle \zeta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ satisfying $0 \leq \zeta_{a_{ij}} + \delta_{a_{ij}} \leq 1$ for all i, j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership,and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

Definition 1.2. [2] A Picture fuzzy matrix (PFM) A of the form, $A = \left(\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ of a non negative real numbers $\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \in [0,1]$ satisfying the condition $0 \leq \zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}} \leq 1$ for all i, j. Where $\zeta_{a_{ij}} \in [0,1]$ is called the degree of membership, $\eta_{a_{ij}} \in [0,1]$ is called the degree of neutral membership and $\delta_{a_{ij}} \in [0,1]$ is called the degree of non-membership.

2. ALGEBRAIC PROPERTIES OF PICTURE FUZZY MATRICES

In this section we define Algebraic operations of picture fuzzy matrices (PFM). Also, we prove some algebraic properties, such as idempotency, commutativity, associativity, absorption law, distributivity and De Morgan's laws over complement.

Now, we are going to define Algebraic operations of Picture fuzzy matrices by restricting the measure of positive, neutral and negative membership but keeping their sum in the interval $[0,1]$.

Definition 2.1. The picture fuzzy matrices A and B of the form, $A = \left(\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle\right)$ and $B = \left(\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \rangle\right)$. Then:

- $A < B$ iff $\forall i, j, \zeta_{a_{ij}} \leq \zeta_{b_{ij}}, \eta_{a_{ij}} \leq \eta_{b_{ij}}$ or $\eta_{a_{ij}} \geq \eta_{b_{ij}}, \delta_{a_{ij}} \geq \delta_{b_{ij}}$
- $A^C = \left(\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}} \rangle\right)$
- $A \vee_{\wp} B = \left(\langle \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) \rangle\right)$
- $A \wedge_{\wp} B = \left(\langle \min\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right) \rangle\right)$

- $A \oplus_{\wp} B = \left(\langle \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \rangle\right)$
- $A \otimes_{\wp} B = \left(\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}} \rangle\right)$.

Definition 2.2. The scalar multiplication operation over PFM A and is defined by $nA = \left(\langle 1 - [1 - \zeta_{a_{ij}}]^n, [\eta_{a_{ij}}]^n, [\delta_{a_{ij}}]^n \rangle\right)$

Definition 2.3. The exponentiation operation over PFM A and is defined by $A^n = \left(\langle [\zeta_{a_{ij}}]^n, 1 - [1 - \eta_{a_{ij}}]^n, 1 - [1 - \delta_{a_{ij}}]^n \rangle\right)$. Let $P_{m\times n}$ denote the set of all the picture fuzzy matrices. The following theorem relation between algebraic sum, and algebraic product of PFMs.

Theorem 2.4. For $A, B \in P_{m\times n}$, then $A \otimes_{\wp} B \leq A \oplus_{\wp} B$.

Proof. Let $A \oplus_{\wp} B = \left(\langle \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \rangle\right)$ and
$A \otimes_{\wp} B = \left(\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}} \rangle\right)$

Assume that,

$\zeta_{a_{ij}}\zeta_{b_{ij}} \leq \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}$

$(i.e)\zeta_{a_{ij}}\zeta_{b_{ij}} - \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}} \geq 0$

$(i.e)\zeta_{a_{ij}}(1 - \zeta_{b_{ij}}) + \zeta_{b_{ij}}(1 - \zeta_{a_{ij}}) \geq 0$

which is true as $0 \leq \zeta_{a_{ij}} \leq 1$ and $0 \leq \zeta_{b_{ij}} \leq 1$

and

$\eta_{a_{ij}}\eta_{b_{ij}} \leq \eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}}$

$(i.e)\eta_{a_{ij}}\eta_{b_{ij}} - \eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}} \geq 0$

$(i.e)\eta_{a_{ij}}(1 - \eta_{b_{ij}}) + \eta_{b_{ij}}(1 - \eta_{a_{ij}}) \geq 0$

which is true as $0 \leq \eta_{a_{ij}} \leq 1$ and $0 \leq \eta_{b_{ij}} \leq 1$

and

$\delta_{a_{ij}}\delta_{b_{ij}} \leq \delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}}$

$(i.e)\delta_{a_{ij}}\delta_{b_{ij}} - \delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}} \geq 0$

$(i.e)\delta_{a_{ij}}(1 - \delta_{b_{ij}}) + \delta_{b_{ij}}(1 - \delta_{a_{ij}}) \geq 0$

which is true as $0 \leq \delta_{a_{ij}} \leq 1$ and $0 \leq \delta_{b_{ij}} \leq 1$

Hence $A \otimes_{\wp} B \leq A \oplus_{\wp} B$.

Theorem 2.5. For $A \in P_{m\times n}$, then

$(i)\ A \oplus_{\wp} A \geq A,$

$(ii)\ A \otimes_{\wp} A \leq A.$

Proof. (i) Let $A \oplus_{\wp} A = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right) \oplus_{\wp} \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$

$A \oplus_{\wp} A = \left(\left\langle 2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^2, (\eta_{a_{ij}})^2, (\delta_{a_{ij}})^2 \right\rangle\right)$

$2\zeta_{a_{ij}} - (\zeta_{a_{ij}})^2 = \zeta_{a_{ij}} + \zeta_{a_{ij}}(1 - \zeta_{a_{ij}}) \geq \zeta_{a_{ij}}$ for all i, j

and $(\eta_{a_{ij}})^2 \leq \eta_{a_{ij}}$ for all i, j

and $(\delta_{a_{ij}})^2 \leq \delta_{a_{ij}}$ for all i, j

Hence $A \oplus_{\wp} A \geq A$.

Similarly, we can prove that (ii) $A \otimes_{\wp} A \leq A$.

Theorem 2.6. For $A, B, C \in P_{m \times n}$, then

$(i)\ A \oplus_{\wp} B = B \oplus_{\wp} A,$

$(ii)\ A \otimes_{\wp} B = B \otimes_{\wp} B,$

$(iii)\ \left(A \oplus_{\wp} B\right) \oplus_{\wp} C = A \oplus_{\wp} \left(B \oplus_{\wp} C\right),$

$(iv)\ (A \otimes_{\wp} B) \otimes_{\wp} C = A \otimes_{\wp} (B \otimes_{\wp} C)$

Proof. (i) Let $A \oplus_{\wp} B = \left(\langle \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \rangle\right)$

$= \left(\langle \zeta_{b_{ij}} + \zeta_{a_{ij}} - \zeta_{b_{ij}}\zeta_{a_{ij}}, \eta_{b_{ij}}\eta_{a_{ij}}, \delta_{b_{ij}}\delta_{a_{ij}} \rangle\right) = B \oplus_{\wp} A.$

(ii) Let $A \otimes_{\wp} B = \left(\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}} \rangle\right)$

$= \left(\langle \zeta_{b_{ij}}\zeta_{a_{ij}}, \eta_{b_{ij}} + \eta_{a_{ij}} - \eta_{b_{ij}}\eta_{a_{ij}}, \delta_{b_{ij}} + \delta_{a_{ij}} - \delta_{b_{ij}}\delta_{a_{ij}} \rangle\right) = B \otimes_{\wp} A.$

(iii) Let $\left(A \oplus_{\wp} B\right) \oplus_{\wp} C = \left(\langle\left(\zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\right) \oplus_{\wp} \left(\zeta_{c_{ij}}, \eta_{c_{ij}}, \delta_{c_{ij}}\right)\rangle\right)$

$= [\left(\zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}\right) + \zeta_{c_{ij}} - \left(\zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}\right)\zeta_{c_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}$

$= \left[\zeta_{a_{ij}} + \zeta_{b_{ij}} + \zeta_{c_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}\zeta_{c_{ij}} - \zeta_{a_{ij}}\zeta_{c_{ij}} - \zeta_{b_{ij}}\zeta_{c_{ij}} + \zeta_{a_{ij}}\zeta_{b_{ij}}\zeta_{c_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\right]$

$= [\zeta_{a_{ij}} + \zeta_{b_{ij}} + \zeta_{c_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{c_{ij}} - \zeta_{b_{ij}}\zeta_{c_{ij}} + \zeta_{a_{ij}}\zeta_{b_{ij}}\zeta_{c_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}]$

(iv) Let $A \oplus_{\wp} \left(B \oplus_{\wp} C\right)$

$= \left[\zeta_{a_{ij}} + \left(\zeta_{b_{ij}} + \zeta_{c_{ij}} - \zeta_{b_{ij}}\zeta_{c_{ij}}\right) - \zeta_{a_{ij}}\left(\zeta_{b_{ij}} + \zeta_{c_{ij}} - \zeta_{b_{ij}}\zeta_{c_{ij}}\right), \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}\right]$

$= [\zeta_{a_{ij}} + \zeta_{b_{ij}} + \zeta_{c_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{c_{ij}} - \zeta_{b_{ij}}\zeta_{c_{ij}} + \zeta_{a_{ij}}\zeta_{b_{ij}}\zeta_{c_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}\eta_{c_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\delta_{c_{ij}}]$

Hence $(A \oplus_{\wp} B) \oplus_{\wp} C = A \oplus_{\wp} (B \oplus_{\wp} C)$

Similarly, we can prove that (iv) $(A \otimes_{\wp} B) \otimes_{\wp} C = A \otimes_{\wp} (B \otimes_{\wp} C)$.

Theorem 2.7. For $A, B \in P_{m\times n}$, then

$(i)\ A \oplus_{\wp} \left(A \otimes_{\wp} B\right) \geq A,$

$(ii)\ A \otimes_{\wp} (A \oplus_{\wp} B) \leq A.$

Proof. (i) Let $A \oplus_{\wp} \left(A \otimes_{\wp} B\right)$

$$= \left(\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \rangle\right) \oplus \left(\left\langle \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$$

$$= [\zeta_{a_{ij}} + \zeta_{a_{ij}}\zeta_{b_{ij}} - \zeta_{a_{ij}}[\zeta_{a_{ij}}\zeta_{b_{ij}}], \eta_{a_{ij}}[\eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}}], \delta_{a_{ij}}[\delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}}]]$$

$$= [\zeta_{a_{ij}} + \zeta_{a_{ij}} + \zeta_{a_{ij}}\zeta_{b_{ij}}[1-\zeta_{a_{ij}}], \eta_{a_{ij}}\left(1-[1-\eta_{a_{ij}}][1-\eta_{b_{ij}}]\right), \delta_{a_{ij}}\left(1-[1-\delta_{a_{ij}}][1-\delta_{b_{ij}}]\right)]$$

$$\geq A.$$

Hence $A \oplus_{\wp} (A \otimes_{\wp} B) \geq A.$

Similarly, we can prove that $(ii) A \otimes_{\wp} (A \oplus_{\wp} B) \leq A$.

The following theorem is obvious.

Theorem 2.8. For $A, B \in P_{m\times n}$, then

$(i)\ A \vee_{\wp} B = B \vee_{\wp} A,$

$(ii)\ A \wedge_{\wp} B = B \wedge_{\wp} A,$

Theorem 2.9. For $A, B, C \in P_{m\times n}$, then

$(i)\ A \oplus_{\wp} \left(B \vee_{\wp} C\right) = \left(A \oplus_{\wp} B\right) \vee_{\wp} \left(A \oplus_{\wp} C\right),$

$(ii)\ A \otimes_{\wp} \left(B \vee_{\wp} C\right) = \left(A \otimes_{\wp} B\right) \vee_{\wp} \left(A \otimes_{\wp} C\right),$

$(iii)\ A \oplus_{\wp} \left(B \wedge_{\wp} C\right) = \left(A \oplus_{\wp} B\right) \wedge_{\wp} \left(A \oplus_{\wp} C\right),$

$(iv)\ A \otimes_{\wp} (B \wedge_{\wp} C) = (A \otimes_{\wp} B) \wedge_{\wp} (A \otimes_{\wp} C).$

Proof. In the following, we shall prove (i), and $(ii)-(iv)$ can be proved analogously.

(i) Let $A \oplus_{\wp} \left(B \vee_{\wp} C\right)$

$$= [\zeta_{a_{ij}} + \max\left(\zeta_{b_{ij}}, \zeta_{c_{ij}}\right) - \zeta_{a_{ij}}.\max\left(\zeta_{b_{ij}}, \zeta_{c_{ij}}\right), \eta_{a_{ij}}.\max\left(\eta_{b_{ij}}, \eta_{c_{ij}}\right), \delta_{a_{ij}}.\max\left(\delta_{b_{ij}}, \delta_{c_{ij}}\right)]$$

$$= [\max\left(\zeta_{a_{ij}} + \zeta_{b_{ij}}, \zeta_{a_{ij}} + \zeta_{c_{ij}}\right) - \max\left(\zeta_{a_{ij}}\zeta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{c_{ij}}\right), \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \eta_{a_{ij}}\eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \delta_{a_{ij}}\delta_{c_{ij}}\right)]$$

$$= [\max\left(\zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \zeta_{a_{ij}} + \zeta_{c_{ij}} - \zeta_{a_{ij}}\zeta_{c_{ij}}\right), \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \eta_{a_{ij}}\eta_{c_{ij}}\right), \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \delta_{a_{ij}}\delta_{c_{ij}}\right)]$$

$$= (A \oplus_{\wp} B) \vee_{\wp} (A \oplus_{\wp} C).$$

Theorem 2.10. For $A, B \in P_{m\times n}$, then

$(i)\ \left(A \wedge_{\wp} B\right) \oplus_{\wp} \left(A \vee_{\wp} B\right) = A \oplus_{\wp} B,$

$(ii)(A \wedge_{\wp} B) \otimes_{\wp} (A \vee_{\wp} B) = A \otimes_{\wp} B,$

$(iii)(A \oplus_{\wp} B) \wedge_{\wp} (A \otimes_{\wp} B) = A \otimes_{\wp} B,$

$(iv)(A \oplus_{\wp} B) \vee_{\wp} (A \otimes_{\wp} B) = A \oplus_{\wp} B.$

Proof. In the following, we shall prove (i), and $(ii)-(iv)$ can be proved analogously.

(i) Let $(A \wedge_{\wp} B) \oplus_{\wp} (A \vee_{\wp} B)$

$= [\min(\zeta_{a_{ij}}, \zeta_{b_{ij}}) + \max(\zeta_{a_{ij}}, \zeta_{b_{ij}})$

$- \min(\zeta_{a_{ij}}, \zeta_{b_{ij}}).\max(\zeta_{a_{ij}}, \zeta_{b_{ij}}), \max(\eta_{a_{ij}}, \eta_{b_{ij}}).\min(\eta_{a_{ij}}, \eta_{b_{ij}}), \max(\delta_{a_{ij}}, \delta_{b_{ij}}).\min(\delta_{a_{ij}}, \delta_{b_{ij}})]$

$= \left(\left\langle \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$

$= A \oplus_{\wp} B.$

In the following theorems, the operator complement obey th De Morgan's laws for the operation $\oplus, \otimes, \vee_{\wp}, \wedge_{\wp}$.

Theorem 2.11. For $A, B \in P_{m\times n}$, then

$(i)\ (A \oplus_{\wp} B)^C = A^C \otimes_{\wp} B^C,$

$(ii)\ (A \otimes_{\wp} B)^C = A^C \oplus_{\wp} B^C,$

$(iii)\ (A \oplus_{\wp} B)^C \le A^C \oplus_{\wp} B^C,$

$(iv)\ (A \otimes_{\wp} B)^C \ge A^C \otimes_{\wp} B^C.$

Proof. We shall prove $(iii), (iv)$, and $(i), (ii)$ are straightforward.

(iii) Let $(A \oplus_{\wp} B)^C = \left(\left\langle \delta_{a_{ij}}\delta_{b_{ij}}, \eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}} \right\rangle\right).$

$A^C \oplus_{\wp} B^C = \left(\left\langle \delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{b_{ij}} \right\rangle\right).$

Since $\delta_{a_{ij}}\delta_{b_{ij}} \le \delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}}$

$\eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}} \ge \eta_{a_{ij}}\eta_{b_{ij}}$

$\zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}} \ge \zeta_{a_{ij}}\zeta_{b_{ij}}$

Hence $(A \oplus_{\wp} B)^C \le A^C \oplus_{\wp} B^C$.

(iv) Let $(A \otimes_{\wp} B)^C = \left(\left\langle \delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}}\zeta_{b_{ij}} \right\rangle\right).$

$A^C \otimes_{\wp} B^C = \left(\left\langle \delta_{a_{ij}}\delta_{b_{ij}}, \eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}}, \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}} \right\rangle\right).$

Since $\delta_{a_{ij}} + \delta_{b_{ij}} - \delta_{a_{ij}}\delta_{b_{ij}} \ge \delta_{a_{ij}}\delta_{b_{ij}}$

$\eta_{a_{ij}}\eta_{b_{ij}} \le \eta_{a_{ij}} + \eta_{b_{ij}} - \eta_{a_{ij}}\eta_{b_{ij}}$

$\zeta_{a_{ij}}\zeta_{b_{ij}} \leq \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}$

Hence $(A \otimes_{\wp} B)^C \geq A^C \otimes_{\wp} B^C$.

Theorem 2.12. For $A, B \in P_{m\times n}$, then

$(i)\ (A^C)^C = A,$

$(ii)\ (A \vee_{\wp} B)^C = A^C \wedge_{\wp} B^C,$

$(iii)\ (A \wedge_{\wp} B)^C = A^C \vee_{\wp} B^C.$

Proof. We shall prove (ii) only, (i) is obvious.

$$A \vee_{\wp} B = \left(\left\langle \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right)\right\rangle\right)$$

$$(A \vee_{\wp} B)^C = \left(\left\langle \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right)\right\rangle\right)$$

$$\Rightarrow A^C = \left(\left\langle \delta_{a_{ij}}, \eta_{a_{ij}}, \zeta_{a_{ij}}\right\rangle\right)$$

$$B^C = \left(\left\langle \delta_{b_{ij}}, \eta_{b_{ij}}, \zeta_{b_{ij}}\right\rangle\right)$$

$$\Rightarrow A^C \wedge_{\wp} B^C = \left(\left\langle \min\left(\delta_{a_{ij}}, \delta_{b_{ij}}\right), \min\left(\eta_{a_{ij}}, \eta_{b_{ij}}\right), \max\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right)\right\rangle\right)$$

Hence $(A \vee_{\wp} B)^C = A^C \wedge_{\wp} B^C$,

Similarly, we can prove that $(iii)(A \wedge_{\wp} B)^C = A^C \vee_{\wp} B^C$.

Based on the Definition 2.1, 2.2 & 2.3., we shall next prove the algebraic properties of Picture fuzzy matrices under the operations of scalar multiplication and exponentiation.

Theorem 2.13. For $A, B \in P_{m\times n}$, then $n > 0$,

$(i)\ n\big(A \oplus_{\wp} B\big) = nA \oplus_{\wp} nB, n > 0,$

$(ii)\ n_1 A \oplus_{\wp} n_2 A = (n_1 + n_2)A, n_1, n_2 > 0,$

$(iii)\ (A \otimes_{\wp} B)^n = A^n \otimes_{\wp} B^n, n > 0,$

$(iv)\ A_1^n \otimes_{\wp} A_2^n = A^{(n_1+n_2)}, n_1, n_2 > 0.$

Proof. For the two PFMs A and B, and $n, n_1, n_2 > 0$, according to definition , we can obtain

(i) Let $n(A \oplus_{\wp} B) = n\left(\left\langle \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}}\right\rangle\right)$

$$= \left(\left\langle 1 - [1 - \zeta_{a_{ij}}]^n [1 - \zeta_{a_{ij}}]^n, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$

$$= \left(\left\langle 1 - [1 - \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}]^n, [\eta_{a_{ij}}\eta_{b_{ij}}]^n, [\delta_{a_{ij}}\delta_{b_{ij}}]^n \right\rangle\right)$$

$$nA \oplus_{\wp} nB = \left(\left\langle\left(1-[1-\zeta_{a_{ij}}]^n,[\eta_{a_{ij}}]^n,[\delta_{a_{ij}}]^n\right) \oplus_{\wp} \left(1-[1-\zeta_{b_{ij}}]^n,[\eta_{b_{ij}}]^n,[\delta_{b_{ij}}]^n\right)\right\rangle\right)$$
$$=[\left(1-[1-\zeta_{a_{ij}}]^n+1-[1-\zeta_{a_{ij}}]^n\right)-\left(1-[1-\zeta_{a_{ij}}]^n\right)\left(1-[1-\zeta_{b_{ij}}]^n\right),[\eta_{a_{ij}}\eta_{b_{ij}}]^n,[\delta_{a_{ij}}\delta_{b_{ij}}]^n]$$
$$=\left(\left\langle 1-[1-\zeta_{a_{ij}}]^n[1-\zeta_{b_{ij}}]^n,[\eta_{a_{ij}}\eta_{b_{ij}}]^n,[\delta_{a_{ij}}\delta_{b_{ij}}]^n\right\rangle\right)$$
$$=\left(\left\langle 1-[1-\zeta_{a_{ij}}+\zeta_{b_{ij}}-\zeta_{a_{ij}}\zeta_{b_{ij}}]^n,[\eta_{a_{ij}}\eta_{b_{ij}}]^n,[\delta_{a_{ij}}\delta_{b_{ij}}]^n\right\rangle\right)$$
$$=n(A \oplus_{\wp} B).$$

(ii) Let $n_1 A \oplus_{\wp} n_2 B = \left(\left\langle\left(1-[1-\zeta_{a_{ij}}]^{n_1},[\eta_{a_{ij}}]^{n_1},[\delta_{a_{ij}}]^{n_1}\right) \oplus_{\wp} \left(1-[1-\zeta_{a_{ij}}]^{n_2},[\eta_{a_{ij}}]^{n_2},[\delta_{a_{ij}}]^{n_2}\right)\right\rangle\right)$

$$=[1-[1-\zeta_{a_{ij}}]^{n_1}+1-[1-\zeta_{a_{ij}}]^{n_2}-\left(1-[1-\zeta_{a_{ij}}]^{n_1}\right)\left(1-[1-\zeta_{a_{ij}}]^{n_2}\right),[\eta_{a_{ij}}]^{n_1}[\eta_{a_{ij}}]^{n_2},[\delta_{a_{ij}}]^{n_1}[\delta_{a_{ij}}]^{n_2}]$$

$$=\left(\left\langle 1-[1-\zeta_{a_{ij}}]^{n_1+n_2},[\eta_{a_{ij}}]^{n_1+n_2},[\delta_{a_{ij}}]^{n_1+n_2}\right\rangle\right)$$
$$=(n_1+n_2)A.$$

(iii) Let $(A \otimes_{\wp} B)^n = [\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, 1-[1-\eta_{a_{ij}}+\eta_{b_{ij}}-\eta_{a_{ij}}\eta_{b_{ij}}]^n, 1-[1-\delta_{a_{ij}}+\delta_{b_{ij}}-\delta_{a_{ij}}\delta_{b_{ij}}]^n]$

$$=[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, 1-[1-\eta_{a_{ij}}]^n[1-\eta_{b_{ij}}]^n, 1-[1-\delta_{a_{ij}}]^n[1-\delta_{b_{ij}}]^n]$$

$A^n \otimes_{\wp} B^n$

$$=[\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, 1-[1-\eta_{a_{ij}}]^n+1-[1-\eta_{b_{ij}}]^n-\left(1-[1-\eta_{a_{ij}}]^n\right)\left(1-[1-\eta_{b_{ij}}]^n\right), 1 - [1-\delta_{a_{ij}}]^n+1-[1-\delta_{b_{ij}}]^n-\left(1-[1-\delta_{a_{ij}}]^n\right)\left(1-[1-\delta_{b_{ij}}]^n\right)]$$
$$=\left(\left\langle\left(\zeta_{a_{ij}}\zeta_{b_{ij}}\right)^n, 1-[1-\eta_{a_{ij}}]^n[1-\eta_{b_{ij}}]^n, 1-[1-\delta_{a_{ij}}]^n[1-\delta_{b_{ij}}]^n\right\rangle\right)$$
$$=(A \otimes_{\wp} B)^n.$$

(iv) Let $A^{n_1} \otimes_{\wp} A^{n_2}$

$$=[\left(\zeta_{a_{ij}}\right)^{n_1+n_2}, 1-[1-\eta_{a_{ij}}]^{n_1}+1-[1-\eta_{a_{ij}}]^{n_2} - \left(1-[1-\eta_{a_{ij}}]^{n_1}\right)\left(1-[1-\eta_{a_{ij}}]^{n-2}\right), 1-[1-\delta_{a_{ij}}]^{n_1}+1 - [1-\delta_{a_{ij}}]^{n-2}-\left(1-[1-\delta_{a_{ij}}]^{n_1}\right)\left(1-[1-\delta_{a_{ij}}]^{n_2}\right)]$$
$$=\left(\left\langle\left(\zeta_{a_{ij}}\right)^{n_1+n_2}, 1-[1-\eta_{a_{ij}}]^{n_1+n_2}, 1-[1-\delta_{a_{ij}}]^{n_1+n_2}\right\rangle\right)$$
$$=A^{(n_1+n_2)}.$$

Hence proved.

Theorem 2.14. For $A, B \in P_{m\times n}$, then $n > 0$,
$(i) nA \le nB, (ii) A^n \le B^n$.

Proof. (i) Let $A \le B \Rightarrow \zeta_{a_{ij}} \le \zeta_{b_{ij}}$ and $\eta_{a_{ij}} \ge \eta_{b_{ij}}$ and $\delta_{a_{ij}} \ge \delta_{b_{ij}}$ for all i, j.
$\Rightarrow 1-[1-\zeta_{a_{ij}}]^n \le 1-[1-\zeta_{b_{ij}}]^n$,
$[\eta_{a_{ij}}]^n \ge [\eta_{b_{ij}}]^n$ and
$[\delta_{a_{ij}}]^n \ge [\delta_{b_{ij}}]^n$. for all i, j.

(ii) Also, $[\zeta_{a_{ij}}]^n \ge [\zeta_{b_{ij}}]^n$,
$1-[1-\eta_{a_{ij}}]^n \le 1-[1-\eta_{b_{ij}}]^n$,
$1-[1-\delta_{a_{ij}}]^n \le 1-[1-\delta_{b_{ij}}]^n$, for all i, j.

Theorem 2.15. For $A, B \in P_{m\times n}$, then $n > 0$,
$(i)\ n(A \wedge_{\wp} B) = nA \wedge_{\wp} nB$,
$(ii)\ n(A \vee_{\wp} B) = nA \vee_{\wp} nB$.

Proof. (i) Let $n(A \wedge_{\wp} B)$

$$= [1-[1-\min\left(\zeta_{a_{ij}}, \zeta_{b_{ij}}\right)]^n, \max\left([\eta_{a_{ij}}]^n, [\eta_{b_{ij}}]^n\right), \max\left([\delta_{a_{ij}}]^n, [\delta_{b_{ij}}]^n\right)]$$
$$= \left[1-\left[\max\left(1-\zeta_{a_{ij}}, 1-\zeta_{b_{ij}}\right)\right]^n, \max\left(\left[\eta_{a_{ij}}\right]^n, [\eta_{b_{ij}}]^n\right), \max\left(\left[\delta_{a_{ij}}\right]^n, [\delta_{b_{ij}}]^n\right)\right]$$
$$= \left[1-\left(\max\left(\left[1-\zeta_{a_{ij}}\right]^n, [1-\zeta_{b_{ij}}]^n\right)\right), \max\left(\left[\eta_{a_{ij}}\right]^n, [\eta_{b_{ij}}]^n\right), \max\left(\left[\delta_{a_{ij}}\right]^n, [\delta_{b_{ij}}]^n\right)\right]$$
$$= \left[\max\left(1-\left[1-\zeta_{a_{ij}}\right]^n, 1-\left[1-\zeta_{b_{ij}}\right]^n\right), \max\left(\left[\eta_{a_{ij}}\right]^n, [\eta_{b_{ij}}]^n\right), \max\left(\left[\delta_{a_{ij}}\right]^n, [\delta_{b_{ij}}]^n\right)\right]$$
$$= nA \wedge_{\wp} nB.$$

Hence $n(A \wedge_{\wp} B) = nA \wedge_{\wp} nB,$
Similarly, we can prove that $(ii) n(A \vee_{\wp} B) = nA \vee_{\wp} nB$.

Theorem 2.16. For $A, B \in P_{m\times n}$, then $n > 0$,
$(i)\ (A \wedge_{\wp} B)^n = A^n \wedge_{\wp} B^n$,
$(ii)\ (A \vee_{\wp} B)^n = A^n \vee_{\wp} B^n$.

Proof. (i) Let $(A \wedge_{\wp} B)^n$

$$= [\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), 1-[\max\left(1-\eta_{a_{ij}}, 1-\eta_{b_{ij}}\right)]^n, 1-[\max\left(1-\delta_{a_{ij}}, 1-\delta_{b_{ij}}\right)]^n]$$

$= [\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), 1-\left(\min\left([1-\eta_{a_{ij}}]^n, [1-\eta_{b_{ij}}]^n\right)\right), 1-\left(\min\left([1-\delta_{a_{ij}}]^n, [1-\delta_{b_{ij}}]^n\right)\right)]$

$= [\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \max\left(1-[1-\eta_{a_{ij}}]^n, 1-[1-\eta_{b_{ij}}]^n\right), \max\left(1-[1-\delta_{a_{ij}}]^n, 1-[1-\delta_{b_{ij}}]^n\right)]$

$A^n \wedge_{\wp} B^n = [\left([\zeta_{a_{ij}}]^n, 1-[1-\eta_{a_{ij}}]^n, 1-[1-\delta_{a_{ij}}]^n\right) \bigcap \left([\zeta_{b_{ij}}]^n, 1-[1-\eta_{b_{ij}}]^n, 1-[1-\delta_{b_{ij}}]^n\right)]$

$= [\min\left([\zeta_{a_{ij}}]^n, [\zeta_{b_{ij}}]^n\right), \max\left(1-[1-\eta_{a_{ij}}]^n, 1-[1-\eta_{b_{ij}}]^n\right), \max\left(1-[1-\delta_{a_{ij}}]^n, 1-[1-\delta_{b_{ij}}]^n\right)]$

$= (A \wedge_{\wp} B)^n$.

Hence $(A \wedge_{\wp} B)^n = A^n \wedge_{\wp} B^n$,

Similarly, we can prove that $(ii) (A \vee_{\wp} B)^n = A^n \vee_{\wp} B^n$.

Theorem 2.17. For $A, B \in P_{m\times n}$, then $n > 0$,
$(A \oplus_{\wp} B)^n \neq A^n \oplus_{\wp} B^n$.

Proof. Let $(A \oplus_{\wp} B)^n$

$= [[\zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}} \zeta_{b_{ij}}]^n, 1-[1-\eta_{a_{ij}} \eta_{b_{ij}}]^n, 1-[1-\delta_{a_{ij}} \delta_{b_{ij}}]^n]$

$A^n = \left(\left\langle [\zeta_{a_{ij}}]^n, 1-[1-\eta_{a_{ij}}]^n, 1-[1-\delta_{a_{ij}}]^n \right\rangle\right)$

$B^n = \left(\left\langle [\zeta_{b_{ij}}]^n, 1-[1-\eta_{b_{ij}}]^n, 1-[1-\delta_{b_{ij}}]^n \right\rangle\right)$

$$A^n \oplus_{\wp} B^n = [[\zeta_{a_{ij}}]^n + [\zeta_{b_{ij}}]^n - [\zeta_{a_{ij}}]^n [\zeta_{b_{ij}}]^n, [1-[1-\eta_{a_{ij}}]^n]^n . [1-[1-\eta_{b_{ij}}]^n]^n, [1 - [1-\delta_{a_{ij}}]^n]^n . [1-[1-\delta_{a_{ij}}]^n]^n]$$

Hence $(A \oplus_{\wp} B)^n \neq A^n \oplus_{\wp} B^n$.

3. NEW OPERATION (@) ON PICTURE FUZZY MATRICES

In this section, we define a new operation(@) on Picture fuzzy matrices and proved their algebraic properties. Further, we discuss the Disstributivity laws in the case where the operations of $\oplus, \otimes, \vee_{\wp}$ and $\wedge_{\wp}$ combined each other.

Definition 3.1. A picture fuzzy matrices A and B of the form, $A = \left(\left\langle \zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}} \right\rangle\right)$ and $B = \left(\left\langle \zeta_{b_{ij}}, \eta_{b_{ij}}, \delta_{b_{ij}} \right\rangle\right)$. Then $A@B = \left(\left\langle \frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2}, \frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2}, \frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2} \right\rangle\right)$.

Obviously, for every two Picture fuzzy matrices A and B, then $A@B$ is a Picture fuzzy matrix.

Simple illustration given: For $A@B$,

$$0 \le \frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2} + \frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2} + \frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2} \le \frac{\zeta_{a_{ij}} + \eta_{a_{ij}} + \delta_{a_{ij}}}{2} + \frac{\zeta_{b_{ij}} + \eta_{b_{ij}} + \delta_{b_{ij}}}{2} \le \frac{1}{2} + \frac{1}{2} = 1.$$

Theorem 3.1. For $A \in P_{m\times n}$, then $A@A = A$.

Proof. Let $A@A = \left(\langle\frac{\zeta_{a_{ij}}+\zeta_{a_{ij}}}{2}, \frac{\eta_{a_{ij}}+\eta_{a_{ij}}}{2}, \frac{\delta_{a_{ij}}+\delta_{a_{ij}}}{2}\rangle\right)$

$= \left(\langle\frac{2\zeta_{a_{ij}}}{2}, \frac{2\eta_{a_{ij}}}{2}, \frac{2\delta_{a_{ij}}}{2}\rangle\right)$

$= \left(\langle\zeta_{a_{ij}}, \eta_{a_{ij}}, \delta_{a_{ij}}\rangle\right) = A.$

Theorem 3.2. For $A, B \in P_{m\times n}$, then

$(i) A@(B \vee_{\wp} C) = (A@B) \vee_{\wp} (A@C),$

$(ii) A@(B \vee_{\wp} C) = (A@B) \vee_{\wp} (A@C).$

Proof. (i) Let $(A@B) \vee_{\wp} (A@C)$

$= [\max\left(\frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2}, \frac{\zeta_{a_{ij}} + \zeta_{c_{ij}}}{2}\right), \min\left(\frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2}, \frac{\eta_{a_{ij}} + \eta_{c_{ij}}}{2}\right), \min\left(\frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2}, \frac{\delta_{a_{ij}} + \delta_{c_{ij}}}{2}\right)]$

$A@(B \vee_{\wp} C)$

$= [\frac{\zeta_{a_{ij}} + \max\left(\zeta_{b_{ij}}, \zeta_{c_{ij}}\right)}{2}, \frac{\eta_{a_{ij}} + \min\left(\eta_{b_{ij}}, \eta_{c_{ij}}\right)}{2}, \frac{\delta_{a_{ij}} + \min\left(\delta_{b_{ij}}, \delta_{c_{ij}}\right)}{2}]$

$= [\max\left(\frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2}, \frac{\zeta_{a_{ij}} + \zeta_{c_{ij}}}{2}\right), \min\left(\frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2}, \frac{\eta_{a_{ij}} + \eta_{c_{ij}}}{2}\right), \min\left(\frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2}, \frac{\delta_{a_{ij}} + \delta_{c_{ij}}}{2}\right)]$

Hence, $A@(B \vee_{\wp} C) = (A@B) \vee_{\wp} (A@C)$

(ii) It can be prove similarly, $A@(B \vee_{\wp} C) = (A@B) \vee_{\wp} (A@C).$

Remark 3.3. For $a, b \in [0,1]$, then $ab \le \frac{a+b}{2}, \frac{a+b}{2} \le a + b - ab.$

Theorem 3.4. For $A, B \in P_{m\times n}$, then

$(i)(A \oplus_{\wp} B) \vee_{\wp} (A@B) = A \oplus_{\wp} B,$

$(ii)(A \otimes_{\wp} B) \wedge_{\wp} (A@B) = A \otimes_{\wp} B,$

$(iii)(A \oplus_{\wp} B) \wedge_{\wp} (A@B) = A@B,$

$(iv)(A \otimes_{\wp} B) \vee_{\wp} (A@B) = A@B.$

Proof. we shall prove (i) and (iii), (ii) and (iv) can be proved analogously.

(i) Let $(A \oplus_{\wp} B) \vee_{\wp} (A@B)$

$$= [\max\left(\zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2}\right), \min\left(\eta_{a_{ij}}\eta_{b_{ij}}, \frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2}\right), \min\left(\delta_{a_{ij}}\delta_{b_{ij}}, \frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2}\right)]$$

$$= \left(\left\langle \zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \eta_{a_{ij}}\eta_{b_{ij}}, \delta_{a_{ij}}\delta_{b_{ij}} \right\rangle\right)$$

$$= A \oplus_{\wp} B.$$

$(iii)(A \oplus_{\wp} B) \wedge_{\wp} (A@B)$

$$= [\min\left(\zeta_{a_{ij}} + \zeta_{b_{ij}} - \zeta_{a_{ij}}\zeta_{b_{ij}}, \frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2}\right), \max\left(\eta_{a_{ij}}\eta_{b_{ij}}, \frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2}\right), \max\left(\delta_{a_{ij}}\delta_{b_{ij}}, \frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2}\right)]$$

$$= \left(\left\langle \frac{\zeta_{a_{ij}} + \zeta_{b_{ij}}}{2}, \frac{\eta_{a_{ij}} + \eta_{b_{ij}}}{2}, \frac{\delta_{a_{ij}} + \delta_{b_{ij}}}{2} \right\rangle\right)$$

$$= A@B,$$

Hence proved.

Result 3.5. The Picture fuzzy matrix forms a semilattice, associativity, commutativity, idempotency under the Picture fuzzy matrix operation of algebraic sum and algebraic product. The distributive law also holds for $\oplus_{\wp}, \otimes_{\wp}$ and $\wedge_{\wp}, \vee_{\wp}, @$ are combined each other.

4. APPLICATIONS

The formation of Picture fuzzy semilattice structure, Picture fuzzy matrix and algebraic structure on this matrix, the results are applicable.

5. CONCLUSIONS

In this paper, we have defined Algebraic operations of Picture fuzzy matrices. We also proved some algebraic properties of Picture fuzzy matrices, such as idempotency, commutativity, associativity, absorption law, distributivity and De Morgan's laws over complement. Finally, we have defined a new operation(@) on Picture fuzzy matrices and discussed distributive laws in the case where the operations of $\oplus_{\wp}, \otimes_{\wp}, \wedge_{\wp}$ and $\vee_{\wp}$ are combined each other. This result can be applied further application of Picture fuzzy matrix theory. For the development of Picture fuzzy semilattice and its algebraic property the results of this paper would be helpful. In the future, the application of the proposed aggregating operators of PFMs needs to be explored in the decision making, risk analysis and many other uncertain and fuzzy environment.

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Biography

I. Silambarasan received his Ph.D. degree in Mathematics from the Department of Mathematics, Annamalai University, Tamil Nadu, India, in 2020. He has published 20 research articles in various international journals (Scopus and Web of Science). He has presented research papers in international conferences and attended many conferences, workshops, and seminars held in India. His research interests are Fuzzy Mathematics.
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