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New type of fuzzy ideals in BCK/BCI algebras

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ABSTRACT

In this paper, we expose pythagorean fuzzy sets in BCK-algebras. Also we define pythagorean fuzzy subalgebra, pythagorean fuzzy ideal in BCK-algebra and investigate some properties of these ideals. Some interesting examples are given. Homomorphism of pythagorean fuzzy set in BCK-algebras are introduce. Moreover we combine the pythagorean fuzzy set and rough sets in BCK-algebras. The concept of rough pythagorean fuzzy ideals in BCK-algebras are introduce.

***Keywords*:** BCK-algebras, Sub algebras, BCK-ideals, Pythagorean fuzzy sets, Pythagorean fuzzy subalgebras, Pythagorean fuzzy ideal, Homomorphism, Rough sets, Rough Fuzzy sets, Rough Pythagorean fuzzy sets, Rough Pythagorean fuzzy ideals

1. INTRODUCTION

The famous fuzzy set was introduced by Zadeh [20] in his classic paper in 1965. The notion of rough set theory was proposed by Z.Pawlak [14]. The concept of rough set theory is an extension of crisp set theory. The theory of rough sets has emerged as another major mathematical approach for managing uncertainty that arises from inexact, noisy or incomplete information. It is turning out to be methodologically significant to the domains of artificial intelligence and cognitive sciences, especially in the representation of reasoning with vague

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and/or imprecise knowledge, data analysis, machine learning, and knowledge discovery [11, 12]. The algebraic approach to rough sets was studied in [8]. Biswas and Nanda [3] introduced the notion of rough subgrougps. Kuroki and Morderson ([9]) discussed the structure of rough sets and rough groups. Kuroki and Wang ([10]) gave some properties of lower and upper approximations with respect to the normal subgroups and the fuzzy normal subgroups. Kuroki [8] introduced the notion of rough ideals in semigroup, which is an extended notion of ideals in semigroups, and gave some properties of such ideals. Dubois and Prade [4] was studied rough fuzzy sets and fuzzy rough sets. Xiao and Zhang [17] established the notion of rough prime ideals and rough fuzzy prime ideals in a semigroup. Iseki [5] introduced two classes of abstract algebras: *BCK*-algebras and *BCI*-algebras. It is known that the class of *BCK*-algebras is a proper subclass of the class of *BCI*-algebras. Lim and Kim [11] introduced the notion of a rough set in *BCK/BCI*-algebras. By introducing the notion of a quick ideal in *BCK/BCI*-algebras, they obtained some relations between quick ideals and upper (lower) rough quick ideals in *BCK/BCI*-algebras. Sun Shin Ahn et al. [16] studied the notion of rough fuzzy ideals in *BCK/BCI*-algebras.

To facilitate our discussion, the remainder of this paper is organized as follows. In section 2 we review some fundamental conceptions of BCK-algebras, fuzzy BCK-algebras, rough sets, pythgorean fuzzy sets, rough fuzzy sets, and rough pythagorean fuzzy sets. In section 3 we propose pythagorean fuzzy subalgebras, pythagorean fuzzy ideal in BCK-algebras. Some examples and important properties of these structures are discussed. In section 4 we introduce the notion of rough pythagorean fuzzy ideals in BCK-algebras (Young Bae Jun *et al*, [21]).

2. FUNDAMENTAL CONCEPTS

This section deals with the fundamental definitions of this work.

A BCK-algebra is a nonempty set Ƀ with a binary operation $*$ and a constant 0 with the following conditions

1. $((n * t) * (n * s)) * (t * s) = 0$
2. $(n * (n * s)) * s = 0$
3. $s * s = 0$
4. $n * s = 0$ and $s * n = 0$ implies $n = s$ for all $n, s, t \in$ Ƀ.

A BCK-algebra is a BCI-algebra if

5. $n \leq s$ implies $n * s = 0$.

In any BCI-algebra Ƀ satisfies the following conditions:

6. $n * s = 0$
7. $(n * s) * t = (n * t) * s$
8. $n \leq s$ implies $n * t \leq s * t$ and $t * s \leq t * n$
9. $(n * s) * (t * s) \leq n * s$.

A nonempty set S of Ƀ is said to be subalgebra of Ƀ if $n * s \in S$ for all $n, s \in S$.

A nonempty set I of Ƀ is called an ideal of Ƀ if (i) $0 \in I$ (ii)$n * s \in I$ and $s \in I$ implies $n \in I$ for all $n, s \in$ Ƀ. A fuzzy set ω of Ƀ is defined by ωω: Ƀ $\rightarrow [0,1]$ and the complement of ω is defined by $\omega^c = 1 - \omega(\mathrm{n})$ for all $n \in$ Ƀ.

A fuzzy subset ω in Ƀ is said to be a fuzzy subalgebra of Ƀ if $\omega(n * s) \geq \min\{\omega(n), \omega(s)\}$. A fuzzy subset ω in Ƀ is called a fuzzy ideal of Ƀ if (i) for all $n \in$ Ƀ, $\omega(0) \geq \omega(n)$ (ii) for all $n, s \in$ Ƀ $\omega(n) \geq min\{\omega(n * s), \omega(s)\}$. A fuzzy subset ω in Ƀ is called a fuzzy bi-ideal of Ƀ if (i) for all $n \in$ Ƀ, $\omega(0) \geq \omega(n)$ (ii) for all $n, s, t \in$ Ƀ $\omega(n * s) \geq min\{\omega(n * s * t), \omega(t)\}$.

Definition 2. 1. [14] Let β be an congruence relation on X. Let Λ be any nonempty subset of X. The sets $\underline{\beta}(\Lambda) = \{x \in X/[x]_\beta \subseteq \Lambda\}$ and $\overline{\beta}(\Lambda) = \{x \in X/[x]_\beta \cap \beta \neq \emptyset\}$ are called the lower and upper approximations of Λ. Then $\beta(\Lambda) = (\underline{\beta}(\Lambda), \overline{\beta}(\Lambda))$ is called rough set in $(X, R) \Leftrightarrow \underline{\beta}(\Lambda) \neq \overline{\beta}(\Lambda)$.

Definition 2. 2. [4] Let β be an congruence relation on X. Let Λ fuzzy subset of X. The upper and lower approximations of Λ defined by

$$\overline{\beta}(\Lambda)(x) = \bigvee_{a \in [x]_\beta} \Lambda(a) \quad \text{and} \quad \underline{\beta}(\Lambda)(x) = \bigwedge_{a \in [x]_\beta} \Lambda(a).$$

$\beta(\Lambda) = (\underline{\beta}(\Lambda), \overline{\beta}(\Lambda))$ is called a rough fuzzy set of Λ with respect to β if $\underline{\beta}(\Lambda) \neq \overline{\beta}(\Lambda)$.

Definition 2. 5. [1] Let X be a nonempty set then an intutionistic fuzzy set can be defined as $\Lambda = \{(x, l_\Lambda(x), m_\Lambda(x))/x \in X\}$ where $l_\Lambda(x)$ *and* $m_\Lambda(x)$ are mapping from X to [0,1] also $0 \leq l_\Lambda(x) \leq 1, 0 \leq m_\Lambda(x) \leq 1, 0 \leq l_\Lambda(x) + m_\Lambda(x) \leq 1$ for all $x \in X$, and represent the degrees of membership and non membership of element $x \in X$ to set X.

Definition 2. 6. [18] Let X be a nonempty set then an Pythagorean fuzzy set can be defined as $\text{þ} = \left\{\left(n, l_{\text{þ}}(n), m_{\text{þ}}(n)\right)/x \in X\right\}$ where $l_{\text{þ}}(n)$ *and* $m_{\text{þ}}(n)$ are mapping from X to [0,1] also $0 \leq l_{\text{þ}}(n) \leq 1, 0 \leq m_{\text{þ}}(n) \leq 1, 0 \leq l^2{}_{\text{þ}}(n) + m^2{}_{\text{þ}}(n) \leq 1$ for all $n \in X$, and represent the degrees of membership and non membership of element $n \in X$ to set X.

Definition 2. 7. [2] Let X be a nonempty set. Let $\text{þ} = \left\{\left(n, \mu_{\text{þ}}(n), \gamma_{\text{þ}}(n)\right)/n \in X\right\}$ be an pythagorean fuzzy set of X. Then an *rpf* set is defined as $\beta(\text{þ}) = (\beta^l(\text{þ}), \beta^u(\text{þ}))$ where

$$\beta^l(\text{þ}) = \{\langle n, \beta^l(l_{\text{þ}}), \beta^l(m_{\text{þ}})\rangle, n \in X\} \text{ and}$$

$$\beta^u(\text{þ}) = \{\langle n, \beta^u(l_{\text{þ}}), \beta^u(m_{\text{þ}})\rangle, n \in X\}$$

with the condition that

$$0 \leq \left(\beta^l(l_{\text{þ}})(n)\right)^2 + \left(\beta^l(m_{\text{þ}})(n)\right)^2 \leq 1, 0 \leq \left(\beta^u(l_{\text{þ}})(n)\right)^2 + \left(\beta^u(m_{\text{þ}})(n)\right)^2 \leq 1.$$

where, $\beta^l(l_{\text{þ}})(n) = \wedge_{n \in [y]_\vartheta} l_{\text{þ}}(y)$ and $\beta^l(m_{\text{þ}})(n) = \vee_{n \in [y]_\vartheta} m_{\text{þ}}(y)$ also,

$\beta^u(l_{\text{þ}})(n) = \vee_{n \in [y]_\vartheta} l_{\text{þ}}(y)$ and $\beta^u(m_{\text{þ}})(n) = \wedge_{n \in [y]_\vartheta} m_{\text{þ}}(y)$.

Example 2. 8.

Let Ƀ $= \{0,1,2,3\}$ be a BCK-algebra with the following table,

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	1	0	1
3	3	3	3	0

Let $\mathfrak{p} = (l_{\mathfrak{p}}, m_{\mathfrak{p}})$ be a *pf* set in Ƀ defined by,

$$l_{\mathfrak{p}}(n) = \begin{cases} .9, & 0 \\ .5, & 1 \\ .4, & 2 \\ .4, & 3 \end{cases} \quad \text{and} \quad m_{\mathfrak{p}}(n) = \begin{cases} .4, & 0 \\ .6, & 1 \\ .5, & 2 \\ .5, & 3 \end{cases} \text{ for all } n \in Ƀ.$$

The equivalence classes of Ƀ are $Ƀ/_{\beta} = \{\{0,1,2\},\{3\}\}$.
Then the lower and upper approximations of $\mathfrak{p}$ are

$$\beta^{l}(l_{\mathfrak{p}})(n) = \begin{cases} .4, & 0 \\ .4, & 1 \\ .4, & 2 \\ .4, & 3 \end{cases} \quad \text{and} \quad \beta^{l}(m_{\mathfrak{p}})(n) = \begin{cases} .6, & 0 \\ .6, & 1 \\ .6, & 2 \\ .5, & 3 \end{cases}$$

Also,

$$\beta^{u}(l_{\mathfrak{p}})(n) = \begin{cases} .9, & 0 \\ .9, & 1 \\ .9, & 2 \\ .4, & 3 \end{cases} \quad \text{and} \quad \beta^{u}(m_{\mathfrak{p}})(n) = \begin{cases} .4, & 0 \\ .4, & 1 \\ .4, & 2 \\ .5, & 3 \end{cases}$$

Then $\beta(\mathfrak{p}) = (\beta^{l}(\mathfrak{p}), \beta^{u}(\mathfrak{p}))$ is a *rpf* set of $\mathfrak{p}$.

3. PYTHAGOREAN FUZZY IDEALS (*PFI*) IN BCK-ALGEBRAS

In this section we introduce the concept of *pfi* in BCK-algebra Ƀ. Also we discuss *pfsa* of Ƀ. Some properties of these ideals and subalgebras are discussed. Illusrate the concept some examples are given.

Definition 3. 1. A *pf* set $\text{þ} = (l_{\text{þ}}, m_{\text{þ}})$ in Ƀ is said to be an *pfsa* of Ƀ if it satisfies:

(B1) $l_{\text{þ}}(n * r) \geq min\{l_{\text{þ}}(n), l_{\text{þ}}(r)\}$

(B2) $\mathrm{m}_{\text{þ}}(n * r) \leq max\{\mathrm{m}_{\text{þ}}(n), \mathrm{m}_{\text{þ}}(r)\}$ for all $n, r \in$ Ƀ.

Example 3. 2.

Let Ƀ $= \{0, \mathrm{A}, \mathrm{E}, \mathrm{I}\}$ be a BCK-algebra with the following table,

*	0	A	E	I
0	0	0	0	0
A	A	0	0	0
E	E	A	0	A
I	I	I	I	0

Let $\text{þ} = (l_{\text{þ}}, m_{\text{þ}})$ be a *pf* set in Ƀ defined by,

$$l_{\text{þ}}(n) = \begin{cases} .9, & 0 \\ .6, & A \\ .4, & E \\ .6, & I \end{cases} \quad \text{and} \quad m_{\text{þ}}(n) = \begin{cases} .3, & 0 \\ .5, & A \\ .7, & E \\ .3, & I \end{cases} \quad \text{for all } n \in \text{Ƀ}.$$

Then þ is a *pfsa* of Ƀ.

Proposition 3. 3.

Every *pfsa* of Ƀ satisfies the inequalities $l_{\text{þ}}(0) \geq l_{\text{þ}}(\mathrm{n})$ and $m_{\text{þ}}(0) \leq m_{\text{þ}}(\mathrm{n})$ for all $n \in$ Ƀ.

Proof: For any $n \in$ Ƀ we have,

$l_{\text{þ}}(0) = l_{\text{þ}}(\mathrm{n} * \mathrm{n}) \geq \min\{l_{\text{þ}}(\mathrm{n}), l_{\text{þ}}(\mathrm{n})\} = l_{\text{þ}}(\mathrm{x})$

$m_{\text{þ}}(0) = m_{\text{þ}}(\mathrm{n} * \mathrm{n}) \leq \max\{m_{\text{þ}}(\mathrm{n}), m_{\text{þ}}(\mathrm{n})\} = m_{\text{þ}}(\mathrm{x})$.

Hence the result.

Definition 3. 4. A *pfs* in Ƀ is said to be a *pfi* of Ƀ if for all $n, r \in$ Ƀ the following conditions are holds:

(B3) $l_{\text{þ}}(0) \geq l_{\text{þ}}(\mathrm{n})$ and $m_{\text{þ}}(0) \leq m_{\text{þ}}(\mathrm{n})$

(B4) $l_{\text{þ}}(n) \geq min\{l_{\text{þ}}(n * r), l_{\text{þ}}(r)\}$

(B5) $\mathrm{m}_{\text{ƥ}}(n) \leq max\{\mathrm{m}_{\text{ƥ}}(n*r), \mathrm{m}_{\text{ƥ}}(r)\}$

Example 3. 5.

Let Ƀ = {0, A, E} be a BCK-algebra with the following table,

*	0	A	E
0	0	0	0
A	A	0	0
E	E	A	0

Let ƥ = $(l_{\text{ƥ}}, m_{\text{ƥ}})$ be a *pf* set in Ƀ defined as follows,

$$l_{\text{ƥ}}(n) = \begin{cases} 1, & n = 0 \\ t_1, & n = A, E \end{cases} \quad \text{and} \quad m_{\text{ƥ}}(n) = \begin{cases} 0, & n = 0 \\ t_2, & n = A, E \end{cases} \quad \text{for all } n \in \text{Ƀ}$$

where $t_1, t_2 \in [0,1]$ also $t_1^2 + t_2^2 \leq 1$.
Then ƥ is a *pfi* in Ƀ.

Lemma 3. 6. Let ƥ be a *pfi* of Ƀ . If the inquality $n * r \leq s$ holds in Ƀ then,

(i) $l_{\text{ƥ}}(n) \geq min\{l_{\text{ƥ}}(r), l_{\text{ƥ}}(s)\}$

(ii) $\mathrm{m}_{\text{ƥ}}(n) \leq max\{\mathrm{m}_{\text{ƥ}}(r), \mathrm{m}_{\text{ƥ}}(s)\}$

Proof: Let $n, r, s \in$ Ƀ such that $n * r \leq s$. Then $(n * r) * s = 0$.

$$\begin{aligned} l_{\text{ƥ}}(n) &\geq min\{l_{\text{ƥ}}(n*r), l_{\text{ƥ}}(r)\} \\ &\geq min\{min\{l_{\text{ƥ}}((n*r)*s), l_{\text{ƥ}}(s)\}, l_{\text{ƥ}}(r)\} \\ &= min\{min\{l_{\text{ƥ}}(0), l_{\text{ƥ}}(s)\}, l_{\text{ƥ}}(r)\} \\ &= min\{l_{\text{ƥ}}(r), l_{\text{ƥ}}(s)\} \end{aligned}$$

Also,

$$\begin{aligned} m_{\text{ƥ}}(n) &\leq max\{m_{\text{ƥ}}(n*r), m_{\text{ƥ}}(r)\} \\ &\leq max\{max\{m_{\text{ƥ}}((n*r)*s), m_{\text{ƥ}}(s)\}, m_{\text{ƥ}}(r)\} \\ &= max\{max\{m_{\text{ƥ}}(0), m_{\text{ƥ}}(s)\}, m_{\text{ƥ}}(r)\} \\ &= max\{m_{\text{ƥ}}(r), m_{\text{ƥ}}(s)\} \end{aligned}$$

Definition 3. 7. A *pfi* in Ƀ is said to be a *pfbi* of Ƀ if for all $n, r, s \in$ Ƀ if the following conditions are holds:

(B3) $l_{þ}(0) \geq l_{þ}(\mathrm{n})$ and $m_{þ}(0) \leq m_{þ}(\mathrm{n})$

(B4) $l_{þ}(n) \geq min\{l_{þ}(n * r * s), l_{þ}(s)\}$

(B5) $\mathrm{m}_{þ}(n) \leq max\{\mathrm{m}_{þ}(n * r * s), \mathrm{m}_{þ}(s)\}$

Example 3. 8.

Let Ƀ $= \{0, \mathrm{x}, \mathrm{y}, \mathrm{z}\}$ be a BCK-algebra with the following table,

*	0	X	Y	Z
0	0	0	0	0
X	X	0	0	X
Y	Y	X	0	Y
Z	Z	Z	Z	0

Let þ $= (l_{þ}, m_{þ})$ be a *pf* set in Ƀ defined by,

$$l_{þ}(n) = \begin{cases} .8, & 0 \\ .5, & X \\ .3, & Y \\ .6, & Z \end{cases} \quad \text{and} \quad m_{þ}(n) = \begin{cases} .3, & 0 \\ .6, & X \\ .7, & Y \\ .3, & Z \end{cases} \quad \text{for all } n \in \text{Ƀ}.$$

Then þ is a *pfbi* of Ƀ.

Lemma 3. 9. Let þ be a *pfbi* of Ƀ . If the inquality $n * r \leq s$ holds in Ƀ then,

(i) $l_{þ}(n * r) \geq min\{l_{þ}(n), l_{þ}(s)\}$

(ii) $\mathrm{m}_{þ}(n * r) \leq max\{\mathrm{m}_{þ}(n), \mathrm{m}_{þ}(s)\}$

Proof: Let $n, r, s \in$ Ƀ such that $n * r \leq s$. Then $(n * r) * s = 0$.

$$l_{þ}(n * r) \geq min\{l_{þ}(n * r * s), l_{þ}(s)\}$$
$$\geq min\{min\{l_{þ}(n), l_{þ}(s)\}, l_{þ}(s)\}$$
$$= min\{l_{þ}(n), l_{þ}(s)\}$$

Also,

$$m_{þ}(n * r) \leq max\{m_{þ}(n * r * s), m_{þ}(s)\}$$
$$\leq max\{max\{m_{þ}(n), m_{þ}(s)\}, m_{þ}(s)\}$$

$= max\{m_{þ}(n), m_{þ}(s)\}$

Lemma 3. 10. Let þ be a *pfi* of Ƀ . If the inquality $n \leq s$ holds in Ƀ then, $l_{þ}(n) \geq l_{þ}(s)$ and $m_{þ}(n) \leq m_{þ}(s)$.

Proof: Let $n, s \in$ Ƀ such that $n \leq s$. Then $n * s = 0$.

$$l_{þ}(n) \geq min\{l_{þ}(n * s), l_{þ}(s)\}$$
$$= min\{l_{þ}(0), l_{þ}(s)\}$$
$$= l_{þ}(s).$$

Also,

$$m_{þ}(n) \geq max\{m_{þ}(n * s), m_{þ}(s)\}$$
$$= max\{m_{þ}(0), m_{þ}(s)\}$$
$$= m_{þ}(s).$$

Theorem 3. 11. If þ is a *pfi* of Ƀ, then for any $n, t_1, t_2 \ldots . t_k \in$ Ƀ, $(\ldots .((n * t_1) * t_2) * \ldots) * t_k = 0$.

Proof: Apply induction on k and also apply Lemma.3.6 and 3.7 we et the result.

Theorem 3. 12. Every *pfi* is a *pfsa* of Ƀ.

Proof: Let þ be a *pfi* of Ƀ. Since $n * r \leq n$ for all $n, r \in$ Ƀ. Then from Lemma.3.7 $l_{þ}(n * r) \geq l_{þ}(n), m_{þ}(n * r) \leq m_{þ}(n)$.

By $l_{þ}(n * r) \geq l_{þ}(n)$

$$\geq min\{l_{þ}(n * r), l_{þ}(r)\}$$
$$\geq min\{l_{þ}(n), l_{þ}(r)\}$$

Also,

$$m_{þ}(n * r) \leq m_{þ}(n)$$
$$\leq max\{l_{þ}(n * r), l_{þ}(r)\}$$
$$\leq max\{l_{þ}(n), l_{þ}(r)\}$$

Hence þ is a *pfsa* of Ƀ.

Converse of Theorem.3.9 is not true. But it is true by applying some condition.

Theorem 3. 13. Let þ be a *pfsa* of Ƀ such that $l_{þ}(n) \geq min\{l_{þ}(s), l_{þ}(t)\}$ and $m_{þ}(n) \leq max\{m_{þ}(s), m_{þ}(t)\}$ for all $n, s, t \in$ Ƀ satisfying thr inequality $n * s \leq t$ then þ is a *pfi* of Ƀ.

Proof: Let ƥ be a *pfsa* of Ƀ. Then by definition $l_{\text{ƥ}}(0) \geq l_{\text{ƥ}}(n)$ and $m_{\text{ƥ}}(0) \leq m_{\text{ƥ}}(n)$ for all $n \in \text{Ƀ}$. Since $n * (n * s) \leq s$, then by the hypothesis, $l_{\text{ƥ}}(n) \geq min\{l_{\text{ƥ}}(n * s), l_{\text{ƥ}}(s)\}$ and $m_{\text{ƥ}}(n) \leq max\{m_{\text{ƥ}}(n * s), m_{\text{ƥ}}(s)\}$. Hence the theorem.

Lemma 3. 14. Let ƥ be a *pfi* of $\text{Ƀ} \Leftrightarrow l_{\text{ƥ}}$ and $m_{\text{ƥ}}^{c}$ are fuzzy ideals of Ƀ.

Proof: Assume that ƥ is a *pfi* of Ƀ. Then obiviously $l_{\text{ƥ}}$ is a *pfi* of Ƀ. Consider for every $n \in \text{Ƀ}$ we have,

$$m_{\text{ƥ}}^{c}(0) = 1 - m_{\text{ƥ}}(0) \geq 1 - m_{\text{ƥ}}(\text{n}) = m_{\text{ƥ}}^{c}(n)$$

Also,

$$\begin{aligned} m_{\text{ƥ}}^{c}(n) &= 1 - m_{\text{ƥ}}(\text{n}) \\ &\geq max\{m_{\text{ƥ}}(n * s), m_{\text{ƥ}}(s)\} \\ &= min\{1 - m_{\text{ƥ}}(n * s), 1 - m_{\text{ƥ}}(s)\} \\ &= min\{m_{\text{ƥ}}^{c}(n * s), m_{\text{ƥ}}^{c}(s)\} \end{aligned}$$

Hence $m_{\text{ƥ}}^{c}$ is a fuzzy ideal of Ƀ.

Conversely, let us take $l_{\text{ƥ}}$ and $m_{\text{ƥ}}^{c}$ are fuzzy ideals of Ƀ then obiviously for every $n \in \text{Ƀ}$ we have $l_{\text{ƥ}}(0) \geq l_{\text{ƥ}}(\text{n})$, $m_{\text{ƥ}}(0) \leq m_{\text{ƥ}}(\text{n})$ and $l_{\text{ƥ}}(n) \geq min\{l_{\text{ƥ}}(n * r), l_{\text{ƥ}}(r)\}$.

We wants to prove $\text{m}_{\text{ƥ}}(n) \leq max\{\text{m}_{\text{ƥ}}(n * r), \text{m}_{\text{ƥ}}(r)\}$.

For that
$$\begin{aligned} 1 - m_{\text{ƥ}}(\text{n}) = m_{\text{ƥ}}^{c}(n) &\geq min\{m_{\text{ƥ}}^{c}(n * s), m_{\text{ƥ}}^{c}(s)\} \\ &= min\{1 - m_{\text{ƥ}}(n * s), 1 - m_{\text{ƥ}}(s)\} \\ &\geq 1 - max\{m_{\text{ƥ}}(n * s), m_{\text{ƥ}}(s)\} \end{aligned}$$

Hence $\text{m}_{\text{ƥ}}(n) \leq max\{\text{m}_{\text{ƥ}}(n * r), \text{m}_{\text{ƥ}}(r)\}$. Thus ƥ be a *pfi* of Ƀ.

Theorem 3. 15. Let ƥ be a *pfi* of $\text{Ƀ} \Leftrightarrow \blacksquare\text{ƥ} = (l_{\text{ƥ}}, l_{\text{ƥ}}^{c})$ and $\blacklozenge\text{ƥ} = (m_{\text{ƥ}}^{c}, \text{m}_{\text{ƥ}})$ are fuzzy ideals of Ƀ.

A mapping $\Theta: \text{Ƀ}_1 \rightarrow \text{Ƀ}_2$ of Ƀ is called a homomorphism if $\Theta(n * t) = \Theta(n) * \Theta(t)$ for all $n, t \in \text{Ƀ}$.

Note: If Θ is a homomorphism of Ƀ then $\Theta(0) = 0$.

Let Θ be a homomorphism of Ƀ. Let ƥ be a *pfs* in Ƀ_2, we define a *pfs* in Ƀ_1 by $\Theta(\text{ƥ}) = (\Theta(l_{\text{ƥ}}), \Theta(\text{m}_{\text{ƥ}}))$ where $\Theta(l_{\text{ƥ}})(n) = l_{\text{ƥ}}(\Theta(n))$ and $\Theta(\text{m}_{\text{ƥ}})(n) = \text{m}_{\text{ƥ}}(\Theta(n))$ for every $n \in \text{Ƀ}$.

Thoerem 3. 16. Let $\Theta: \text{Ƀ}_1 \rightarrow \text{Ƀ}_2$ be a homomorphism of Ƀ. If ƥ is a *pfi* in Ƀ_2 then $\Theta(\text{ƥ})$ is a *pfi* in Ƀ_1.

Proof: For all $n \in Ƀ_1$ we have,

$$\Theta(l_{ҏ})(n) = l_{ҏ}(\Theta(n)) \leq l_{ҏ}(0) = l_{ҏ}(\Theta(0)) = \Theta(l_{ҏ})(0)$$

$$\Theta(m_{ҏ})(n) = m_{ҏ}(\Theta(n)) \geq m_{ҏ}(0) = m_{ҏ}(\Theta(0)) = \Theta(m_{ҏ})(0)$$

Let $n, t \in Ƀ_1$. Then,

$$min\{\Theta(l_{ҏ})(n * r), \Theta(l_{ҏ})(r)\} = min\{l_{ҏ}(\Theta(n * r)), l_{ҏ}(\Theta(r))\}$$
$$= min\{l_{ҏ}(\Theta(n) * \Theta(r)), l_{ҏ}(\Theta(r))\}$$
$$\leq l_{ҏ}(\Theta(n)) = \Theta(l_{ҏ})(n).$$

Also,

$$max\{\Theta(m_{ҏ})(n * r), \Theta(m_{ҏ})(r)\} = max\{l_{ҏ}(\Theta(n * r)), l_{ҏ}(\Theta(r))\}$$
$$= min\{l_{ҏ}(\Theta(n) * \Theta(r)), l_{ҏ}(\Theta(r))\}$$
$$\leq l_{ҏ}(\Theta(n)) = \Theta(l_{ҏ})(n).$$

Converse of the Theorem.3.13 may not be true. But if we apply a condition on Θ it is true.

Theorem 3. 17. Let $\Theta: Ƀ_1 \to Ƀ_2$ be an epimorphism of Θ and let ҏ be a *pfs* in $Ƀ_2$. If Θ(ҏ) is *pfi* of $Ƀ_1$, then ҏ is *pfi* in $Ƀ_2$.

Proof: For any $n \in Ƀ_2$there exist $i \in Ƀ_1$suchthat $\Theta(i) = n$.
Then

$$l_{ҏ}(n) = l_{ҏ}(\Theta(n)) = \Theta(l_{ҏ})(n) \leq \Theta(l_{ҏ})(0) = l_{ҏ}(\Theta(0)) = l_{ҏ}(0).$$

$$m_{ҏ}(n) = m_{ҏ}(\Theta(n)) = \Theta(m_{ҏ})(n) \leq \Theta(m_{ҏ})(0) = m_{ҏ}(\Theta(0)) = m_{ҏ}(0).$$

Let $n, t \in Ƀ_2$ then there exist $i, j \in Ƀ_1$ $\Theta(i) = n$ and $\Theta(j) = t$.
Then,

$$l_{ҏ}(n) = l_{ҏ}(\Theta(n)) = \Theta(l_{ҏ})(i)$$
$$\geq min\{\Theta(l_{ҏ})(i * j), \Theta(l_{ҏ})(j)\}$$
$$= min\{l_{ҏ}(\Theta(i * j)), l_{ҏ}(\Theta(j))\}$$
$$= min\{l_{ҏ}(\Theta(i) * \Theta(j)), l_{ҏ}(\Theta(j))\}$$
$$= min\{l_{ҏ}(n * t), l_{ҏ}(t)\}$$

$$m_{ҏ}(n) = m_{ҏ}(\Theta(n)) = \Theta(m_{ҏ})(i)$$
$$\leq max\{\Theta(m_{ҏ})(i * j), \Theta(m_{ҏ})(j)\}$$
$$= max\{m_{ҏ}(\Theta(i * j)), m_{ҏ}(\Theta(j))\}$$
$$= max\{m_{ҏ}(\Theta(i) * \Theta(j)), m_{ҏ}(\Theta(j))\}$$

$= max\{m_{þ}(n * t), m_{þ}(t)\}$

Hence the theorem.

4. APPROXIMATIONS OF PYTHAGOREAN FUZZY IDEALS (*pfi*) IN BCK-ALGEBRAS

This section deals with approximations of *pfs* in *BCK*-algebras. Also we introduce the lower and upper approximations of *pfs* in Ƀ. We prove the lower and upper approximations of *pfi* is also a pfi of Ƀ.

Theorem 4. 1. Let Ɓ be a congruence relation on Ƀ. If þ is a *pfi* of Ƀ then $Ɓ^u$ is a *pfi* of Ƀ.

Proof: Since þ is a *pfi* of Ƀ, then by the definition

$l_{þ}(0) \geq l_{þ}(n)$ and $m_{þ}(0) \leq m_{þ}(n)$.

Also,

$$Ɓ^u(l_{þ})(0) = \vee_{s\in[0]_{Ɓ}} l_{þ}(0)$$

$$\geq \vee_{s\in[0]_{Ɓ}} l_{þ}(t)$$

$$= Ɓ^u(l_{þ})(n)$$

and

$$Ɓ^u(m_{þ})(0) = \wedge_{s\in[0]_{Ɓ}} m_{þ}(0)$$

$$\leq \wedge_{s\in[0]_{Ɓ}} m_{þ}(t)$$

$$= Ɓ^u(m_{þ})(n)$$

Then for any $n, t \in$ Ƀ we have,

$$Ɓ^u(l_{þ})(n) = \vee_{s\in[n]_{Ɓ}} l_{þ}(n)$$

$$\geq \vee_{p*q\in[n]_{Ɓ}*[t]_{Ɓ}, q\in[t]_{Ɓ}} \{min\{l_{þ}(p * q), l_{þ}(q)\}\}$$

$$\geq \vee_{p*q\in[n*t]_{Ɓ}, q\in[t]_{Ɓ}} \{min\{l_{þ}(p * q), l_{þ}(q)\}\}$$

$$\geq min\{\vee_{p*q\in[n*t]_{Ɓ}} l_{þ}(p * q), \vee_{q\in[t]_{Ɓ}} l_{þ}(p)\}$$

$$= min\{Ɓ^u(l_{þ})(n * t), Ɓ^u(l_{þ})(t)\}$$

Also,

$$Ɓ^u(m_{þ})(n) = \wedge_{s\in[n]_{Ɓ}} m_{þ}(n)$$

$$\leq \wedge_{p*q\in[n]_{Ɓ}*[t]_{Ɓ}, q\in[t]_{Ɓ}} \{max\{m_{þ}(p * q), m_{þ}(q)\}\}$$

$$\leq \wedge_{p*q\in[n*t]_{\theta},q\in[t]_{\theta}} \left\{max\{m_{\text{þ}}(p*q), m_{\text{þ}}(q)\}\right\}$$
$$\leq max\{\wedge_{p*q\in[n*t]_{\theta}} m_{\text{þ}}(p*q), \wedge_{q\in[t]_{\theta}} m_{\text{þ}}(p)\}$$
$$= max\{\theta^{u}(m_{\text{þ}})(n*t), \theta^{u}(m_{\text{þ}})(t)\}$$

Hence $\theta^{u}(\text{þ}) = (\theta^{u}(l_{\text{þ}}), \theta^{u}(m_{\text{þ}}))$ is a pfi of Ƀ.

Theorem 4. 2. Let θ be a congruence relation on Ƀ. If þ is a *pfi* of Ƀ then θ^{l} is, if it is nonempty, a *pfi* of Ƀ.

Proof: Similar to Theorem.4.3.

Note: Let þ be a pf subset of Ƀ and let $(\theta^{l}(\text{þ}), \theta^{u}(\text{þ}))$ be rpf subset of Ƀ. If $\theta^{l}(\text{þ})$ and $\theta^{u}(\text{þ})$ are pfi of Ƀ, then $(\theta^{l}(\text{þ}), \theta^{u}(\text{þ}))$ a rpfi of Ƀ.

Corollary 4. 3. If þ is a pfi of Ƀ then $(\theta^{l}(\text{þ}), \theta^{u}(\text{þ}))$ is a rpfi of Ƀ.

5. CONCLUSIONS

In this paper, we have pyhthagorean fuzzy sets in BCK-algebras. Also pythagorean fuzzy subalgebras and pythgorean fuzzy ideals in BCK-algebras are defined. Some examples are investigated. Moreover rough pythagorean fuzzy ideals in BCK-algebras defined. Some interesting properties of these ideals are proved.

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Biography

P. Dhanalakshmi doing her Ph.D. in Mathematics from the Department of Mathematics, Annamalai University, Tamil Nadu, India. She has published 8 research articles in various international journals (Scopus and Web of Science). She has presented research papers in international conferences and attended many conferences, workshops, and seminars held in India. Her research interests are Fuzzy algebra and rough set theory.

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