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Quaternionic division by zero is implemented as multiplication by infinity in 4D hyperspace

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ABSTRACT

Quaternionic division by zero is implemented as an algebraic multiplication by infinity performed in a 4D multispatial hyperspace. The operation is shown to be possible when it is executed within abstract multispatial hyperspace comprising several pairs of quasigeometric spaces (or quasispatial structures), which are paired as primary and its dual reciprocal space.

***Keywords*:** Quaternionic division by zero, multiplication by infinity, multispatiality

1. INTRODUCTION

The feasibility of an abstract algebraic division by zero has already been demonstrated in [1] under the assumption that real numbers can be identified with geometric points of an abstract spacetime. Transition from mere feasibility to actual possibility, however, required us to somehow reconcile the fact that while abstract numbers are usually (i.e. if considered in terms of their values) not regarded as depending on the basis in which their set-theoretical algebraic space is represented, the points of quasigeometric 4D spacetime are represented in heterogeneous basis. Notice that when bases of two representations are both homogeneous and alike, then they can be safely omitted. If one basis is homogeneous whereas the other basis is heterogeneous however, such bases of representations cannot be neglected anymore.

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Abstract integral basis was defined in the context of algebraic numbers in [2]. Although useful in pure mathematics, such bases do not convey the message that I want to emphasize, namely that traditional mathematics operates only within unspoken confines of single-space reality (SSR). The unspoken SSR paradigm caused numerous problems that are not always perceivable in mathematics, but often caused confusion in physical applications. Hence, we need a more general basis that could be applicable in realistic physical theories. However, some previously unanticipated experimental results hint at the possibility that the physical (and thus also geometric) reality we live in resembles a multispatial structure and therefore requires a paradigm shift from the SSR paradigm to a multispatial reality (MSR) paradigm, under which all algebraic operations should be unrestricted [3]. It has already been shown that division by zero becomes nonissue in both the real [1] and complex domain [3] if the MSR paradigm is adopted. Now we should extrapolate the algebraic division by zero onto quaternions whose domain incorporates both scalar, quasiscalar, and vector fields.

2. ALTERNATIVE EXPLANATIONS OF RESTRICTED DIVISION BY ZERO

After establishing realistic division by zero in the complex domain one may still wonder whether some nontraditional approaches to division by zero can retain some merit.

An unusual division by zero was proposed in several papers [133-136]

$$\frac{0}{0}=\frac{1}{0}=\frac{z}{0}=0\ without\ specifying\ the\ \frac{bases}{spaces}\ in\ which\ the\ zeros\ belong \qquad (0)$$

with numerous examples of the unusual operation. It has been suggested in [3] that for this operation to be realistic one needs some specific restrictions as to domains of its validity. It is not a call to restrict the operations but to avoid its unintended consequences.

Without some kind of operational restrictions imposing contingency on validity of the proposal offered in the formula (0) could be viewed as just replacement of the traditional prohibition on division by zero by an implicit prohibition on multiplication by zero, because multiplying the chain of equations (0) by zero would equate 0=1=z where z is an arbitrary complex number or function of complex variable.

One way to avoid the above dilemma could be via establishing the following inversions: $0 \to 1$, $1 \to 0$, $\infty \to \infty$ [137] p. 39. Note that the effect of an inversion is to change a cross ratio into its conjugate [137] p. 44. But all the inversions were discussed under the unspoken SSR paradigm. Complex inversion transformation w = 1/z is discussed in [138]. Also inverse images of vector bundles and differential 1-forms are concisely discussed in [139].

Though some authors view the components of hypercomplex numbers as homogeneous algebraic or projective coordinates of points in 3D spaces [140], this view emerged from the unwarranted and indefensible SSR paradigm. But under the MSR paradigm all compound hypercomplex numbers are heterogeneous and thus cannot be represented in a single space.

I do not see how the proposal (0) could be justified under the old SSR paradigm where there is only one single space and the space has only one zero. Note that set-theoretical or algebraic or topological spaces are merely sets, which can be equated with certain selection rules, not really spaces in purely geometrical sense.

For in geometrically meaningful spaces numbers are identified with geometric points and intervals, and – since the latter depend on particular basis – the reciprocal points could not belong in the same space without crushing its proper native basis. Therefore, if every 3D geometric space should have its own homogeneous basis, then one needs to design a certain multispatial hyperspace comprising quasigeometric twin pairs of dual reciprocal 4D spaces. Each such 4D pair should have heterogeneous basis, however.

Under the MSR paradigm, however, the proposal (0) could be admissible, provided each zero is qualified by the basis of the space to which it belongs. It is because each space of the pair of primary and reciprocal spaces that compose the given pair of 4D multispatial hyperspace, has its own local zero, each quite distinct one. But the zero that resides in the reciprocal space appears as infinity in its primary space of the given pair. For without being explicitly qualified by the space or the native basis to which each of the zeros standing in the formula (0) belong, the formula may cause conceptual confusion. The formula (0) could be made true indeed, but no truth is electable, no matter how many examples support it.

3. GEOMETRIC VS. ALGEBRAIC FEATURES HINT AT MULTISPATIALITY

In mathematics quaternions are regarded as generalizations of the complex numbers, and rightly so. Although hyperbolic functions can be treated in imaginary/complex terms alone, they could be entertained exclusively in the real domain too, as Vincenzo Riccati did (in 18th century) by formal analogy to trigonometric functions with the use of geometry alone, i.e. considering only hyperbolic functions without ever diving into the purely algebraic theory of complex numbers that employs imaginary unit, with which he was obviously familiar [4].

Just by looking at the mindboggling algebraic feature of the two hyperbolic functions:

$$2\cosh(x) = e^{x} + e^{-x} \text{ and } 2\sinh(x) = e^{x} - e^{-x} \tag{1}$$

being represented as just compound real functions of the increasing exponential e^{x} and the decreasing exponential e^{-x} [4], one may wonder whether or not there could still emerge yet another story to be told (once it is discovered and understood) about the structural/geometric characteristics of complex representations of these two functions.

Today we usually ascribe the modern representations of complex numbers to Euler, and presumably rightly so, but the fact that these two hyperbolic functions are doubled was so striking to me that I have asked myself where is the other space that permits such apparent doubling, despite the fact that the hyperbolic functions contain the decreasing term e^{-x}. Were it not for a certain interspatial reciprocity the doubling would not be possible under the SSR paradigm. The algebraic Eulerian representation of the hyperbolic sine function written as

$$\sinh(x) = \frac{e^{x} - e^{-x}}{2i} \tag{2}$$

completes the purely geometric quest [5]. It also answers my question by implying that the function is not just imaginary but reciprocal as well. This is fully in sync with physics. The argument that a complex quantity can never be observed is true, but it would be inaccurate to conclude that all physical quantities that appear in a theory should be real [6].

I am not trying here to elevate the genius of V. Riccati at the expense of that of Euler. I admire them both. Yet comparison of the geometric approach of Riccati with the algebraic approach of Euler indicates that development of mathematics is not just sequentially lined up.

From a purely abstract synthetic point of view these two approaches should be entirely equivalent. For if something is really true, it could be explained in many different ways.

Although the formula (2) is more elaborate than that of Riccati, it did not answer the very question what it means geometrically. Despite the fact that the eq. (2) can be depicted in the complex plane, comparison of the Riccati's formula (1) with the Eulerian eq. (2) implies the quite realistic possibility that multispatial geometric structure may really exist. Note that tensor calculus cast within the confines of the SSR paradigm can create somewhat false assurance of completeness rather than providing unquestionably valid representations.

Case in point: Levi-Civita once declared that change of direction during parallel shift of a vector depends on the end points alone for it is independent of a particular selection of the line connecting the end points (which we can assume to be elements of a geodesic); hence such shift has always absolute character and consequently also the change of the vector's direction has an absolute character [7]. But the fact is that the vector being shifted can have a twist (according to the Frenet-Serret formulas of differential geometry [8]). Hence such abstract reasonings – correct as they might appear – do not always guarantee the validity of derived conclusions, which also depend on understanding of the issues involved therein.

I am not criticizing any particular scientist but the presumptuous arrogance of the unduly influential great minds who sometimes impose their own views on others to the detriment of prospective development of the other sciences that blindly accept unbalanced mathematical conclusions. This is not about mere convenient formulations or the use of particular words but about blatant suppressions of truths – even if it is inadvertent – because the conclusion discredited the actual possibility of presence of extra nonradial repulsive effects of the usual radial/center-bound force fields [9] as well as some consequences thereof [10,11].

Having contrasted the two functions with abstract duality principle (which in hindsight seems easy, because V. Riccati lived before its discovery) I have realized that we need two reciprocal spaces (or at least quasispatial structures) in order to get fully geometrical nature of the complex numbers and – by extension – quaternions and even octonions as the highest possible hypercomplex numbers [12]. The reciprocal feature of spatiality was also stumbled upon by Lagrange [13]. However, the dogma of traditional mathematics is that geometric dimensionality of spatial structures could be arbitrarily increased via enumeration of higher dimensions, which is not true [14]. One can enumerate dimensions, but this does not mean that the higher than fourth dimensions expand the basis [13,14], Here I am using the new notion of reciprocity as it could be used in the context of physical applications. For purely algebraic, group-theoretical usage of reciprocity in the conventional set-theoretical context consult [15] and references therein. However, the conventional notion of reciprocity was cast in the framework of the unspoken SSR paradigm of abstract mathematical reality.

The need of espousing the MSR paradigm along with the dual twin spaces (primary and its reciprocal) is virtually implicit in theoretical treatment of vector fields flowing in tubes. See [16]: The solenoidal vector field is necessarily continuous, so that a vector tube cannot terminate abruptly in a cross section of finite area inside the field. Thus, a vector tube must be closed or originate from a point and terminate in like manner. It is not possible for this latter case to occur within the field because the fact that the intensity of the tube is constant implies that vector field itself must become infinite as the cross section of the tube shrinks to zero [16] – see also [3,17]. Thus, if a vector tube is not closed it must terminate on the boundary of the space occupied by the field or move off to infinity [16]. This fact becomes nonissue if the reciprocal space – which represents the actual infinity – is introduced [3].

The issue of heterogeneous bases coexisting within the same quasigeometric structure has already been discussed – mainly in operational rather than structural/geometric terms, though – in [13,14]. Henceforth one could try to reconsider the persistently nagging issue of the unwarranted traditional prohibition on division by zero in quaternionic domain too.

It was asserted that application of imaginary numbers does not explain their character [18]. Early applications were algebraic. Geometric representation of the complex numbers allowed further generalization of the idea of number [19]. Quaternion is an operator which changes one vector into another [20]. Yet every quaternion has vector and scalar parts [21].

Due to opposition to the use of quaternions by leading mathematicians, Gibbs defined vector algebra, in which nD vector may be defined as an ordered n-tuple of real numbers [22]. Algebraic vector (i.e. abstract n-tuple) is the basic quantity of Grassmann's algebra of extensions, whereas vector appears as just a subsidiary part in Hamilton's quaternions [23], [24]. Tait once remarked that quaternions do have slight theoretical advantage over vectors, because quaternion product retain associativity, whereas the Gibbsian vector product does not, for $j \times (k \times k) = 0 \neq (j \times k) \times k = - j$ in the basis (1,j,k,l) [25].

The realistic possibility of performing unambiguously the algebraic division by zero has been shown in the domain of real numbers as well as in the domain of complex numbers in [3]. By realistic I mean one that is unambiguous, both operationally and structurally correct, and one that is both executable and constructible as well. Now the division by zero shall be extended onto the quaternionic numbers domain. Reciprocal quaternion was defined in [26].

In order to be realistic, mathematical procedures must fit the abstract structures in which they operate. The so-called synthetic way was already used by Isaac Newton [27] and then other mathematicians after him. I use my (new) synthetic approach to solving problems in both mathematics and physical sciences with the aid of geometries, experimental hints and some theoretical thought experiments too. The essence of my new synthetic approach is the presumption that algebraic/operational procedures should correspond to geometric and/or quasigeometric structures to ensure the validity of both of them. This feature was already also emphasized by Newton who realized their codependence [28]. The qualifier 'new' is intended to distinguish it from numerous previous synthetic approaches to mathematics. Yet the syntheses rely on theoretical hints supplied by actual or virtual/theoretical experiments. The hints are supposed to guide the development of mathematical theories.

4. VARIOUS FEATURES AND INTERPRETATIONS OF QUATERNIONS

At first Hamilton was primarily interested in algebraic fields equipped with a general multidimensional multiplication [29]. This led to his discovery of quaternions. He concluded that it is not possible to construct a system of complex numbers furnished with only three units in which division is possible [30]. Quaternions represent the operation of compounding rotations in the regular Euclidean space [31,32].

Since quaternions are always given by a scalar (even if the scalar happen to be zero) and a pseudovector, it is clear that the identification of a pure quaternion with 3D vector cannot be general [33]. One may also say that quaternion comprises biscalar and bivector because imaginary scalar is a biscalar and imaginary vector is bivector [34].

Quaternion is a line segment originating in the null point of a 4D coordinate system and ending in a point (x,y,z,w) analog to the concept of vector in a 3D space and thus an abstract multiplication by a quaternion represents rotation in 4D space [35] with the radius r acting as an enlargement [36]. A quaternion manifold is a 4m-dimensional real manifold carrying an integrable almost quaternal structure [37]. Translations are represented by sum p+q of two quaternions p and q and rotations by the products: pq, qp, etc [38].

The real quaternions form a linear associative algebra over the real numbers having as basis four independent elements [39] but the basis is not unique [40]. A quaternion algebra over a field F is a central simple F-algebra of degree 2. If the characteristic of F is different from 2, it can be shown that every 4D quaternion algebra Q has the basis (1,j,k,l) ... called quaternion basis with jk = l = -jk and jj = -1 [41]; compare also [42]. Hence the scalar part of a quaternion can be no longer identified with the imaginary unit [43]. The tensor/scalar/value is thus a quaternion that has turn zero – it extends only [44].

By operations on right lines in a 2D plane the imaginary unit received a perfectly clear interpretation, as denoting a second unit line, at right angles to that line which had been selected to represent positive unity [45]. Just as scalar is a quotient of two parallel vectors, quaternion is a quotient of two (inclined) vectors [46,48]. Vector is pure quaternion (with zero scalar part) [49,50]. Geometry and physics require thus a scheme to deal with fields of quantities such as vectors, tensors and spinors [51].

5. GENERAL 3D ROTATIONS REQUIRE 4D QUATERNIONS

Quaternions form real vector space [52]. Quaternion functions are thus also 4D analytic functions [53]. If understood as rotations, quaternions could be used for 4-periodic analytic functions of two complex variables [54,55]. Analytic function of 4D quaternion variables can also be defined with reference to differential equations and so hyperanalytic functions are also defined by extension [56].

It is known that in hypercomplex setting nullpoint can represent singularity [57]. An example of an analytic regular quaternion function composed of two analytic functions of complex variables shows that in general, singularities are not really isolated points since in this case they fill out 2D manifold [58]. Recall that two vectors are orthogonal to each other if their symmetric bilinear form equals zero [59].

Hurwitz defined inversion of quaternions in terms of values [60] and has shown that any rotation in 4D space could be expressed in the form $q \rightarrow pq/r$ where q is the quaternion and p,r, are each unit quaternions [61]. Every 3×3 orthogonal matrix is either a rotation matrix or else a rotation matrix plus a reflection in the plane of the rotation and every 3×3 rotation matrix corresponds to a quaternion [62]; compare also [63,64]. Since linear metric can be made compatible with Clifford-valued differential forms [65], it is possible to construct a matrix representation of any Clifford algebra and to extend the very construction to any Cartesian prooduct of classical matrix domains [66].

Rotation is a product of reflections in two planes whereas reflection in a line is a rotation round the line [67]. Also screws are actually compound rotations given by 4×4 matrix of det = 1 [68,69]. Note that the resultant of two successive rotations or turns (i.e., the sum of two arc steps) is commutative only when the arcs are cocircular [70]. For any vector **v** and a quaternion q whose complex conjugate is q* vector rotation is qvq* whereas frame rotation is q*$\mathbf{v}$q [71]. Thus, the vectors of which quaternion is the quotient or product are actually perpendicular to each other [72,73]. In the sense outer product of a point and a vector is to be interpreted as a rotor [74]. And quaternionic derivatives were discussed in [75,76].

Quaternions and the based upon them hyperbolic rotations could be helpful also in relativistic theories [77] including wave theories and field theories [78,79]. Minkowski was too hasty in disclaiming the adaptability of Hamilton's quaternions in physics [80].

Quaternionic expression of the special-relativistic transformation q' = QqQ is

$$Q = \frac{1}{\sqrt{2}}\left(\sqrt{1+\gamma} + \mathbf{u}\sqrt{1-\gamma}\right) \tag{3}$$

where $\mathbf{u}$ is a unit vector in the direction of motion of s' relatively to s. Observe that the

$$\gamma = \sqrt{1 - \left(\frac{v}{c}\right)^2} > 0 \tag{4}$$

is the special-relativistic factor so that the vector of Q is imaginary while its scalar part is real [81]. This demonstrates that quaternions are not incompatible with special relativity.

Quaternions offer also simple and elegant description of spin of a single particle, perhaps superior to that of the conventional quantum mechanics [82]. However, the expansion of abstract 4D geometric and quasigeometric structures should be understood first for in the quaternionic setting the scalar potential is represented in the dual reciprocal space.

An analytical representation of functions of quaternion variables similar to complex-analytical treatment is presented in [83,84]. A 4D abstract quaternion gradient operator was defined also in [85]. Nonetheless, all heretofore treatment of 4D quaternions and other 4D spatial and quasispatial structures assumed the former unspoken SSR paradigm, which is prone to conceptual and/or logical errors, because the SSR paradigm is quite indefensible.

By therorem of Frobenius: The only division algebras over the real field are the real $\mathbb{R}$, complex $\mathbb{C}$, and $\mathbb{H}$, the Hamilton quaternions [86]. All nonassociative quaternion algebras are thus division algebras [87]. One of the most remarkable properties of quaternion group is that each of its subgroups is self-conjugate [88]. Also notice that triquaternion – as a sum of three quaternions – represents a transformation by similitude [89].

Recall that despite the Hurwitz's theorem that was interpreted as limiting hypercomplex numbers to 4D quaternions and 8D octonions Pfister proved that one could extend product of two sums of 2^n squares for every n [90]. This achievement proved not only that no proof is final but also that mathematics is restricted not only by imagination of mathematicians or the lack thereof but also depends on one's knowledge of other related topics.

It has already been shown that complementary hidden variables can be revealed just by extending the operational features of differential calculus irrespective of the interpretation of variables and/or parameters involved [91]. There are 16 distributively generated near-rings [92]. The quaternion group is of order 8 [93]. Quaternionic structure in $\mathbb{R}^4$ and an octonionic structure in $\mathbb{R}^8$ is discussed in [94]; compare also [95-97].

Nonetheless, two quaternions involve 8 parameters, while for a general 4D rotation only 6 parameters are required [98]. This is also perceivable in quite another way [14]. Since in biquaternion quantum mechanics Bell inequality is violated [99], which makes it desirable also for physics. Hence despite several mathematical idiosyncrasies, quaternions appear as entities being of significant physical importance. Thus, before jumping from quaternions to higher-dimensional generalizations we should explore 4D quasispatial structures first.

Hamilton realized that a quaternion q may be reduced to the form $q = r (\cos\alpha + i\sin\alpha)$ of a complex number [100], for quaternions are pairs of complex numbers [101]. This fact can also be viewed as follows: If p is a unit-length vector and v is any vector perpendicular to p, then by multiplying v on the left by the quaternion $q = \cos\alpha + p \sin\alpha$ we rotate it about p through the angle α [102]. Complex number in a plane $x + iy = r \exp(i\theta)$ and a rotation of the angle α around the z axis can be represented by $\exp(i\alpha)$ we get $\exp(i\alpha) (x + iy) = r \exp i(\alpha + \theta)$; hence similar to rotations in a 2D plane, a rotation about an axis passing through the origin [i.e. through local zero] and parallel to a given unitary vector by an angle α can be obtained by taking real quaternion transformation giving a complexified quaternion [103].

Since quaternion and its reciprocal are commutative [104], the quaternionic divisions by zero is consistent with the operational structure of quaternions. Moreover, the operations agree with the supposition that the primary and reciprocal spaces are just two views on the same quaternionic object, which suggests that the operational and geometric structures of quaternions are compatible indeed. This assertion is also supported by the fact that each of division algebras (quaternions included [105]) corresponds to some vector space [106] p. 61.

Quaternion algebra H is viewed as a 4D real R-algebra defined on the Hamilton's table with e = (1,0,0,0), j = (0,1,0,0), k = (0,0,1,0), l = (0,0,0,1) so that we get $= j^2 = k^2 = l^2 = jkl = -e$. The Hamiltonian algebra is a noncommutative and associative division algebra so the 3D space Rj + Rk + Rl = Im H is imaginary vector subspace [107] p.197. Hence for two 3D vectors u,v, we obtain $uv = -e<u,v> + u \times v$ where the $<u,v>$ denotes their scalar product [107] p. 199.

Algebraically, quaternion has the form $Q = a_0I + iA$ where A is a 3-vector and a_0 is a real and I is the scalar unit; the quaternion is also a product of 3-vectors A and B of the form

$$Q = AB = \frac{1}{2}\,(AB + BA)I + i[\frac{1}{2i}\,(AB - BA)] = (A\bullet B)I + i(A\times B) \qquad (5)$$

as it was presented in [108,109]. In geometric setting geometric product of two 3-vectors the quaternion can be also written in operational terms [110,111] as geometric product:

$$AB = A\bullet B + A\wedge B = A\bullet B + iA\times B \qquad (6).$$

The time-space duality apparent in tachyons was extensively discussed in the context of quaternions in [112]. The facts speak in favor of the possibility of actual presence of a certain 4D timespace that corresponds to the usual 4D spacetime as its dual space [13,14]. Dual quaternion frames and their relation to Frenet-Serret formulas were discussed with the aid of dual quaternions in [113]; compare also [114,115]. Since proper evaluation of trajectories with the use of Frenet-Serret formulas of differential geometry [8] did lead to the theoretical discovery of certain nonradial effects of purely radial/center-bound force fields [9,10,117,118] that were already confirmed in experiments [11,116, we should not ignore these ideas, for they require altering the former mathematics [91].

The so-called differential 4-nabla operators were operationally defined simply by a sort of theoretical extrapolation primarily for electrodynamics and, relativity theories in various textbooks [119], and even in the context of quaternions in [120,121]. Although not quite inadmissible for most practical purposes, they can perpetuate nonsenses and thus can cause disaster often resulting in some operational confusion, because their unspoken underlying assumption was – and often still is – that dimensions of geometric spaces can be added like piling up potatoes *ad infinitum* in a heap-space. This is evidently a process of enumeration not really construction. Traditional mathematics created a lot of conceptual confusion.

This infantile assumption permeates both mathematics and physics despite the fact that it tacitly violates numerous otherwise unquestionable achievements of such great minds as Lagrange, Abel and Galois [13] and their – generally suppressed – abstract geometrical consequences [14]. One consequence of suppressing inconvenient truth that the great minds stated – even though the truth was not understood even by them – is the routine tacit denial of existence of multispatial reality (MSR). The usual assumption was that we live in a single space reality (SSR), which is allegedly so self-evident that often it is not even mentioned in the former pure mathematics, after which physics blindly followed.

This denial of the multispatial structure of the mathematical and physical reality (and of the true character of operators that are supposed to operate on various objects defined and immersed in the unmentioned reality) was in sync with many other conceptual nonsenses perpetrated in former mathematics, the chief of which was the now infamous prohibition on division by zero. Once the latter nonsensical obstacle was removed and division by zero is admitted in domains of real numbers [1] and complex numbers [3], one can see that the SSR paradigm is unacceptable in general and thus should be replaced with the MSR.

6. DIVISION OF ZERO BY ZERO IN QUATERNIONIC DOMAIN

Quaternionic zero evaluates to 0 + j0 + k0 +l 0 i.e. to four ordered zeros. From the formula $0 \cdot \infty = 1$ [1] we get $0 \cdot \mathrm{i}\infty = \mathrm{i}$ [3] and consequently also $0 \cdot \mathrm{j}\infty = \mathrm{j}$. Thus we can obtain

$$[\mathrm{R}]\frac{0,0,0,0}{0,0,0,0} = [\mathrm{R}](0 + \mathrm{j}0 + \mathrm{k}0 + \mathrm{l}0) \cdot [\mathrm{ʁ}]\left(\frac{1}{\infty} + \frac{1}{\mathrm{j}\infty} + \frac{1}{\mathrm{k}\infty} + \frac{1}{\mathrm{l}\infty}\right) = [\mathrm{R}]2(\mathrm{j} + \mathrm{k} + \mathrm{l} - 1) \tag{7}$$

which is an abstract division of quaternionic zero by quaternionic zero that is implemented via multiplication of the quaternionic zero represented in the primary set-theoretical space by infinity depicted as reciprocal inverse quaternionic zero represented in the associated (or paired) space that is reciprocal to the primary space. Abstract multispatial structures that are relevant to the chain of equations (7) were discussed in [3]. Relation between the primary and reciprocal vectors was discussed in dyadic terms in [122] pp. 24,74; see dyads also in [123]. Recall that conjugate quaternion is conventionally defined as 0-j0-k0-l0 [124] and therefore the result (7) can be rewritten in terms of negative conjugate quaternion.

Since the usual imaginary operator i can operate on any realistic spaces I am using the quaternion frame depicted as (1,j,k,l) in order to avoid misunderstandings with other bases. The meaning of the results shall be discussed elsewhere. One can easily check the chain of equations (7) by comparing it with formulas developed and concisely presented in [125].

The values of the two quaternionic bases are: [R] = [1,1,1,1] pertaining to the primary space, and [ʁ] = [1,1,1,1] that pertains to the reciprocal space; their values are incidentally the same because reciprocal of unity is unity too: 1/1 = 1. But the two bases are denominated differently, for if the primary basis is denominated in meters, for instance, then the basis of its reciprocal space is denominated in inverse meters; for more elaborate explanation of the meanings and applications of the pair of primary and reciprocal bases see [3].

Notice that I am using the small dot to indicate that the above multiplication spans two quite distinct spaces, namely the primary space and its reciprocal one. Each of these spaces has its own native basis and the abstract algebraic bases designating every pair of these two mutually reciprocal spaces are also mutually inverted, i.e. mutually reciprocal too [3]. Note that the neutral [quasiscalar or biscalar] element in the quaternionic basis is foreign because it does not belong in the 3D [Euclidean] space with [the native vector] basis (i,j,k) [126].

The basis of the reciprocal space is foreign with respect to the primary space and thus one must convert the reciprocal basis into the basis of the primary space (and consequently also the objects that are represented within the reciprocal basis should be converted to their representations within the primary basis [R] [3]) before attempting any operations on them in the primary space (7) and vice versa. In other words: we can operate on either space, but all the operands must be depicted in the native basis of the space chosen for the operations.

We should not mix operands depicted in distinct bases. If one operand is represented in the native basis of the given space while the other operand is represented in its foreign basis, the advanced algebraic operation of addition or multiplication (or subtraction or division) on them should not proceed without prior conversion to native basis and native representation. For such incompatible operations could breed some conceptual and/or logical nonsenses.

The duality of reciprocal spaces resembles conceptually the duality of contravariant and covariant representations of objects whose concise abstract discussion is presented in [127]. Nevertheless, almost all ideas of our traditional mathematics have been developed upon the unspoken assumption of the SSR paradigm. Although not entirely incorrect, the old ideas often create false impressions of completeness of the representations based upon the ideas.

Moreover, one can calculate also few other relevant combinations of divisions by zero

$$[\text{ʀ}]\frac{0,0,0,0}{0,0,0,0} = [\text{ʀ}](0 + \mathrm{j}0 + \mathrm{k}0 + \mathrm{l}0)\ \cdot [\text{ʁ}]\left(\frac{1}{\infty} - \frac{1}{\mathrm{j}\infty} - \frac{1}{\mathrm{k}\infty} - \frac{1}{\mathrm{l}\infty}\right) = [\text{ʀ}]3 \qquad (8)$$

$$[\text{ʀ}]\frac{\overline{0,0,0,0}}{0,0,0,0} = [\text{ʀ}](0 - \mathrm{j}0 - \mathrm{k}0 - \mathrm{l}0)\ \cdot [\text{ʁ}]\left(\frac{1}{\infty} + \frac{1}{\mathrm{j}\infty} + \frac{1}{\mathrm{k}\infty} + \frac{1}{\mathrm{l}\infty}\right) = [\text{ʀ}]3 \qquad (9)$$

which involve two conjugate quaternionic divisions by zero via multiplication by infinity.

Division by zero of conjugate zero-quaternions via multiplication by infinity gives

$$[\text{ʀ}]\frac{\overline{0,0,0,0}}{\overline{0,0,0,0}} = [\text{ʀ}](-2)(1 + \mathrm{j} + \mathrm{k} + \mathrm{l}) \qquad (10)$$

which is the associated division involving conjugate zero-quaternions.

The reader can see that division of quaternionic zero by real zero can be performed as

$$[\text{ʀ}]\frac{0,0,0,0}{0} = [\text{ʀ}](0 + \mathrm{j}0 + \mathrm{k}0 + \mathrm{l}0) \cdot [\text{ʁ}]\left(\frac{1}{\infty}\right) = [\text{ʀ}](1 + \mathrm{j} + \mathrm{k} + \mathrm{l}) = [\text{ʀ}]\frac{0}{(0,0,0,0)} \qquad (11)$$

which is remarkable indeed because the result is unit quaternion regardless of whether the zero-quaternion stands in the numerator or the denominator of the operation of division.

The resulted unit quaternion caps the algebraic completeness of 4D representation. One can also check the n-th root of unity in quaternion space, which is concisely discussed in [128].

Operational role of the dual reciprocal space introduced to house the set-points-numbers that can be expanding to infinity, which represent the geometric actual infinity [3] as a local notion, is to permit operations on infinity. The latter operations require operators turning the set-numbers representing the reciprocal space/infinity into single-number to be operated on.

7. DUAL RECIPROCAL MULTISPATIAL DIFFERENTIAL OPERATORS

Physics should guide synthetic reasonings in mathematics. Since both mass and energy refer to the same physical entity, [both potential energy that is usually being represented as the reciprocal length interval V = 1/r and] mass can also be viewed as an additional spacelike component of a higher-dimensional vector [106] p. 177.

If the primary space is denominated in meters or seconds, then its reciprocal space can be denominated in inverse meters [1/m] or in inverse seconds (cycles) [1/s], respectively. That is why I have introduced the heterogeneous 4D nabla operator $\boxed{\nabla}$ that can operate at once on geometric objects depicted in two distinct spaces (i.e. the primary and its reciprocal space), each of which has its own distinct native basis not just for vectors but for numbers as well [3,14]. Recall that the primary and reciprocal spaces are actually just distinct views on the same geometric or quasigeometric object whereas sets are merely different selection rules, not really spaces. Former mathematics used spaces and sets interchangeably, though.

If the new heterogeneous 4D nabla operator $\boxed{\nabla}$ is defined symbolically on Grassmann's terms as an abstract complex (rather than real) primary extension of the regular 3D nabla operator ∇ via a certain new 1D furled (i.e dimensionally compressed) nabla operator $\forall$:

$$\boxed{\nabla} \coloneqq \nabla + i\forall \tag{12}$$

which obviously represents (symbolically the prototype of) the Helmholtz's theorem [129], then according to conventional algebraic principles the sqared formula (12) should read as:

$$\boxed{\nabla}^2 \coloneqq \nabla^2 - \forall^2 + 2i\nabla\forall \tag{13}$$

where the curved nabla $\forall$ is an imaginary operator with respect to the regular nabla operator ∇, for $\forall$ cannot be directly represented in terms of the vector basis of the primary space on which the regular 3D nabla operator ∇ is defined. For no any other (than those three already present in the 3D nabla operator) distinct fourth component could be directly orthogonal to all the other three directional components of the 3D nabla operator [3,14]. Although the Helmholtz theorem is still valid within any single 3D vector field under the SSR paradigm, when the symbolic prototype is squared under the MSR paradigm then its consequences are much wider. Proofs are merely derivations from the axioms and primitive notions assumed as true under the currently espoused paradigms. Proofs do not guarantee truth of theorems.

Note that the Helmholtz's theorem corresponds to Poisson equation pertaining to radial fields alone [130,131]. The expanded formula (13) suggests necessity of presence of some nonradial effects of radial/center-bound force fields, which have already been discovered [8-

10,116]. Although the Poisson equation could be extended onto few extra radial components as sources of the fields under the SSR paradigm too, the extension suggested by the eq. (13) under the MSR paradigm suggests expansion onto some reciprocal components, related to terms depending on the reciprocal scalar radial potential V = 1/r. These nonradial reciprocal terms have already been experimentally confirmed [11,116] and they conform to Frenet-Serret formulas [8]. Thus former mathematics must be altered. It is not an option.

The furled 1D nabla operator $\forall$ has thus the effect of reducing the formal dimensional representation from 3D to 1D. It supports thus unfurling of coordinates in the primary space and furling of coordinates in the reciprocal space associated with the primary one [3]. Note that the regular imaginary unit i performs the role of an interspatial operator that operates on spatial/geometric and quasispatial structures, and the differential operators are defined on these structures, whereas the imaginary units that form the quaternion basis are (j,k,l).

Yet one can see that if the speed of light c is used as the reciprocal/inversed interspatial conversion coefficient from the 3D space represented by the usual 3D nabla ∇ into (one of) its reciprocal/inverse 1D spaces represented by the furled/curly nabla operator $\forall$, then

$$\boxed{\nabla}^2 \coloneqq \nabla^2 + (\mathrm{i}\forall)^2 \quad \Rightarrow \quad (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - (cdt)^2 \tag{14}$$

which implies also the reformulated line element cast within the 4D multispatial hyperspace. Here the 1D curly nabla operator determines the special-relativistic temporal term: $c\forall t = cdt$. One can see that if the spatial differentials dx,dy,dy would be treated in vectorial terms rather than scalar values then the algebraic signature (+++-) of the eq. (14) could be turned into the reversed geometric signature (---+) [3]. These two topics shall be further discussed more comprehensively elsewhere in physical setting on relativistic terms.

We do not doubt that the squared line element ds standing on the RHS of the implication (14) is quite correct in the sense that its consequences produce numerous conceptually valid and experimentally confirmed results obtained in various relativistic theories of physics. Yet according to the fully evaluated prototype formula (13) the squared line element should read

$$\boxed{\nabla}^2_{\mathrm{ST}} \Leftrightarrow \ (\mathrm{ds})^2 = (\mathrm{dx})^2 + (\mathrm{dy})^2 + (\mathrm{dz})^2 - (\mathrm{cdt})^2 + 2\mathrm{ic}(\mathrm{dx} + \mathrm{dy} + \mathrm{dz})\mathrm{dt} \tag{15}$$

which includes an imaginary extra term on the far RHS that is qualified by the interspatial imaginary unit i. Hence the extra term cannot belong to the same abstract 4D structure of spacetime ST as the traditional line element standing in (14). The variable s in the arc length ds refers to the trajectory curve – as it is commonly used in differential geometry – whereas the subscript ST in the compound 4D operator $\boxed{\nabla}$ pertains to the 4D spacetime.

This expanded line element (15) does not mean that the conventional line element (14) is incorrect but that the unspoken SSR paradigm does not really reflect the true nature of both the abstract mathematical and the experiential physical reality, which – if expanded according to the MSR paradigm – may reveal its different operational and structural nature indeed. The MSR paradigm reveals features unanticipated under the old SSR paradigm.

Notice that the multispatial expansion of the fully evaluated line element is depicted in the units of as if elapsing length L variable, because the speed of light c virtually eliminates the usual elapsing time parameter t. Therefore, we could depict the imaginary interspatial expansion and then write the squared expansion component $(dL)^2$ of the line element ds as

$$(\mathrm{dL})^2 \coloneqq \mathrm{c}(\mathrm{dx}+\mathrm{dy}+\mathrm{dz})\mathrm{dt} = (\mathrm{dx}+\mathrm{dy}+\mathrm{dz})\mathrm{dl} \tag{16}$$

where the speed of light c = (dl/dt) is assumed as possibly varying as it does in physical fields containing matter in the form of either material substance or energy. Recall that the speed of light remains constant only in empty space devoid of any material substance, not in general.

Yet any speed should be measured in units denominated in native basis of the given space. The expansion formula (16) looks like a combined linearly spreading spatial component whose square value dL^2 = (dx + dy + dz)dl eliminates the impact of the temporal parameter t that is usually associated with steadily elapsing time. The remarkable fact is that the regular spreading component of the expanding 3D length interval (dx + dy + dz) is not squared but is simply linearly decomposed vector, which is evidently spreading in yet another imaginary space (or a quasispatial structure) to be fully identified elsewhere.

The other component dl of the squared expansion element dL^2 is apparently not really spreading in the other space, however. In the sense the expansion element dL ties the dual reciprocal spatial representation of the fully evaluated line element ds depicted in (15). In the sense the differential dl represents the single-valued approximation of infinity, which is set-valued number residing in the furled reciprocal 1D space associated with its 3D primary space [3]. Recall that 3D spatial spreading is the feature of 3D spaces for only they permit making extension of the form (dx + dy + dz). One cannot spread and still be moving along 1D line, because spreading aside and moving along single 1D line is contradiction in terms. Note that we can introduce abstract quaternionic principle of duality of points and planes by using the same vector symbol to denote either a point or a plane [132], which – if it is to be consistently implemented – would require at least one set of dual reciprocal pairs of spaces: spacetime ST and timespace TS, which henceforth shall also be indicated by subscript TS.

Synthesizing (with the use of abstract duality principle underlying the MSR paradigm) the actual logical and conceptual contradictions existing under the SSR paradigm I propose

$$\boxed{\nabla}^2_{\mathrm{TS}} \Leftrightarrow \ (\mathrm{d}\text{ŧ})^2 = (\mathrm{do})^2 + (\mathrm{dp})^2 + (\mathrm{dq})^2 - \left(\frac{1}{\mathrm{c}}\mathrm{ds}\right)^2 + 2\mathrm{i}\frac{1}{\mathrm{c}}(\mathrm{do}+\mathrm{dp}+\mathrm{dq})\mathrm{ds} \tag{17}$$

yet another 4D differential operator that is dual to the spatial 4D differential operator that is depicted in (15) and is operating on the quasispatial structure of 4D spacetime ST.

This symbolic 4D differential operator that operates on the 4D structure of timespace TS is just another view of the usual 4D spacetime ST [13] equipped with heterogeneous native vectorlike 4D basis is depicted above as (o,p,q,s). The homogeneous temporal 3D part of the basis is (o,p,q). The subscript TS standing by the 4D operator $\boxed{\nabla}$ in (17) pertains to the 4D timespace TS which appears as an abstract dual reciprocal counterpart of the 4D spacetime. Since both the ST and TS reference the same underlying 4D quasispatial structure [14], the 3D time variable ŧ – which is quite distinct but not much different than the elapsing time parameter t that does not vary within the LBS. The ŧ corresponds to the 3D length variable s. It should also be decomposable into certain mutually orthogonal three temporal dimensions in its own native 3D time-based space (TBS) just as length/distance intervals that determine the trajectory s are routinely decomposed within the usual 3D Euclidean length-based space (LBS), when the temporal intervals are treated as 3D vectors.

Let me recapitulate the point: Since both the 4D quasispatial structures (spacetime and timespace) comprise regular Euclidean 3D spaces I shall call the native 3D primary space, in which the distances s live and vary, the length-based space (LBS), and the other native 3D primary space, in which the elapsing time variable ŧ could now be decomposed and thus can vary its temporal components quite independently of each other, the time-based space (TBS). The 3D elapsing time variable ŧ that lives in the TBS is not the same as the regular elapsing time parameter t commonly being used within the LBS. These and other related conceptual issues shall be further discussed elsewhere in a more enhanced context.

Analogously to eq. (16) we could deduce the temporal imaginary interspatial expansion and write the squared expansion component dT of the expanding temporal line element dŧ

$$(\mathrm{dT})^2 \coloneqq \frac{1}{\mathrm{c}}(\mathrm{do}+\mathrm{dp}+\mathrm{dq})\mathrm{ds} = (\mathrm{do}+\mathrm{dp}+\mathrm{dq})\mathrm{d}\mathcal{t} \qquad (18)$$

where $\mathcal{t}$ denotes time-based parameter corresponding to the length variable that lives and varies in the reciprocal 1D space LBS. Here the inverted speed of light $c^{-1} = (d\mathcal{t}/ds)$ is shown in the native temporal basis of the TBS that is different than the native basis of the LBS, The temporal expansion component dT is thus analogous to the length expansion component dL.

Thus the abstract dual analogy between the eq. (16) and (18) is formally quite complete. Nonetheless, the temporal basis is depicted along the trajectory ds of the LBS but not in the primary basis that is native to the length-based homogeneous frame/basis (x,y,z) of the LBS.

Therefore, the temporal variable $\mathcal{t}$ of the TBS is not the same as the usual elapsing time parameter t routinely used within the LBS. The squared $dT^2 = (do + dp + dq)d\mathcal{t}$ elevates the squared temporal expansion element dT^2 while eliminating the usual trajectory arc ds that is depicted in the native homogeneous basis (x,y,z) of the 3D LBS for the arc ds cannot be cast directly into the (foreign to it) native homogeneous temporal 3D basis (o,p,q) that is native to the temporal space TBS in which the dT is represented natively.

The extra imaginary components are abstract hinges on which the two 3D spaces, LBS and TBS, hinge within these two (dual reciprocal) 4D quasispatial structures, i.e. spacetime (ST) and timespace (TS). These issues cannot be comprehensively discussed in the present, essentially purely algebraic context. Nevertheless, we can show that even the traditional mainly algebraic theory of 4D quaternions apparently already embraced some aspects of the emphasized above abstract 3D-4D connections/expansions.

The whole issue of multispatial expansions and interspatial connections does not exist in the context of the SSR paradigm. They are features emerging naturally in the context of the MSR paradigm though. These and related issues shall be further discussed elsewhere.

8. MULTISPATIAL DIFFERENTIAL OPERATORS FOR 4D QUATERNIONS

Multispatial hyperspace has a heterogeneous basis comprising both the native and the foreign bases of the hyperspace. This creates a conceptual dilemma that must be resolved because all operands involved in each operation should be represented within the same basis. Although we just cannot turn a heterogenous basis into a homogeneous one without violating some operational or structural principles, having two dual spatial or quasispatial structures we can devise an interspatial operation within the given multispatial hyperspace.

Denoting algebraic representation of any geometrical or quasigeometrical object in the native basis of the given space by η and the representation of the same object in the foreign basis of the dual reciprocal space by ϕ then the symbolic prototype (13) can be rewritten as

$$\boxed{\nabla}^2(\eta \oplus i\phi) \coloneqq \nabla^2\eta - \not{\nabla}^2\phi + 2i\nabla\eta \otimes \not{\nabla}\phi \tag{19}$$

where the symbol $\oplus$ denotes heterogeneous addition of these two terms represented in the two distinct bases (i.e. η in the native basis and ϕ in its foreign basis) whereas the symbol $\otimes$ denotes a certain interspatial multiplication of the two (both dimensionally- as well as basisweise-different) differential nabla operators, one of which is depicted in the native basis of the primary space and the other in its foreign basis. Compare also [122] p. 59ff.

The interspatial operation of multiplication would require thus conversions of variables and parameters to be used as operands into native basis of the space being operated on. At this point the operation is not yet defined. We know, however, that we had to use either the one or the other basis, in which we have formulated first the length-based squared extension dL^2 in (16) and then the other, temporal squared extension dT^2 in (18).

Both these interspatial extensions are always imaginary and thus are invisible in the real domain. In the above notation the heterogeneous addition appears as an imaginary Coriolis-like extra acceleration term [141]. Although the term is usually depicted as negative (i.e. with reversed direction) and real rather than imaginary, these are not its essential features. The eq. (19) is mathematically more general. Its physics shall be discussed elsewhere.

Recall that [R] = [1,1,1,1] is the native basis of the primary space, and [ʁ] = [1,1,1,1] is the native reciprocal basis of the reciprocal space while being foreign basis with respect to the primary space [3] and that the reciprocal derivatives are not the inverses of the primary derivatives of functions when three variables are considered [122] pp. 29,53. The same is true of the differential operators that depict derivatives and differentials in general.

Hence from the eq. (15) of spacetime and the eq. (16) of length-based spatial extension we obtain an alternative 4D spacetime-length (STL) representation of the 4D spacetime as

$$\boxed{\nabla}^2_{STL} \Leftrightarrow (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - (cdt)^2 + 2i\,(dx + dy + dz)dl \tag{20}$$

or

$$\boxed{\nabla}^2_{STL} \Leftrightarrow (ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - (cdt)^2 + 2i\,(dL)^2 \tag{21}$$

which can be rewritten via Laplacian $\nabla^2\mathbf{S}$ and divergence Div $\mathbf{S} = \nabla\bullet\mathbf{S}$ of the interval/vector $\mathbf{S}$

$$\boxed{\nabla}^2_{STL} \Leftrightarrow (ds)^2 = \nabla^2\mathbf{S} - c^2\not{\nabla}^2 t + 2i\,(\nabla \bullet \mathbf{S}) \cdot \not{\nabla} l \tag{22}$$

or

$$\boxed{\nabla}^2_{STL} \Leftrightarrow (ds)^2 = \nabla^2\mathbf{S} - c^2\mathrm{Dif}^2 t + 2i\,\mathrm{Div}\,\mathbf{S}\ \cdot \mathrm{Dif}^1 l \tag{23}$$

where the formula is written also in terms of foreign differential operators. Here the foreign differential 1D spatial interval $\not{\nabla} l = dl = \mathrm{Dif}^1 l$ while the 3D spatial interval is $\mathbf{S} = \mathbf{x} + \mathbf{y} + \mathbf{z}$. Recall that Laplacian of a scalar function of a scalar is usually defined as ∇^2 = DivGrad [142] but it could be defined for a vector function as well – compare [143,161].

While from the eq. (17) of timespace and the eq. (18) of time-based temporal expansion we obtain an alternative 4D timespace-duration (TSD) representation of the 4D timespace as

$$\boxed{\nabla}^2_{\mathrm{TSD}} \Leftrightarrow (\mathrm{d}\text{ŧ})^2 = (\mathrm{do})^2 + (\mathrm{dp})^2 + (\mathrm{dq})^2 - \left(\frac{1}{\mathrm{c}}\mathrm{ds}\right)^2 + 2\mathrm{i}\,(\mathrm{do} + \mathrm{dp} + \mathrm{dq})\mathrm{d}\textit{ŧ} \tag{24}$$

or

$$\boxed{\nabla}^2_{\mathrm{TSD}} \Leftrightarrow (\mathrm{d}\text{ŧ})^2 = (\mathrm{do})^2 + (\mathrm{dp})^2 + (\mathrm{dq})^2 - \left(\frac{1}{\mathrm{c}}\mathrm{ds}\right)^2 + 2\mathrm{i}\,(\mathrm{dT})^2 \tag{25}$$

or

$$\boxed{\nabla}^2_{\mathrm{TSD}} \Leftrightarrow (\mathrm{d}\text{ŧ})^2 = \nabla^2\text{Ŧ} - \frac{1}{\mathrm{c}^2}\not\nabla^2 \mathrm{l} + 2\mathrm{i}\,(\nabla \bullet \text{Ŧ}) \cdot \not\nabla\textit{ŧ} \tag{26}$$

or

$$\boxed{\nabla}^2_{\mathrm{TSD}} \Leftrightarrow (\mathrm{d}\text{ŧ})^2 = \nabla^2\text{Ŧ} - \frac{1}{\mathrm{c}^2}\mathrm{Dif}^2 \mathrm{l} + 2\mathrm{i}\,\mathrm{Div}\,\text{Ŧ} \cdot \mathrm{Dif}^1\textit{ŧ} \tag{27}$$

which indicate presence of formal analogy between the abstract quasispatial 4D structures of the usual 4D spacetime and the corresponding to it 4D timespace.

Hence, it is evident that the varying abstract temporal line segment/element/arc ŧ (that corresponds to the spatial, i.e. the distance- or just length-based line segment/element/arc s) could be decomposed within the temporal basis to form the temporal interval Ŧ just as s forms the length-based interval S. Similarly, the temporal inverted/reciprocal 1D interval *ŧ* corresponds to the 1D spatial (i.e. length-based) inverted/reciprocal interval l.

Although existence of some complementary hidden variables is always possible [91], the importance of the present paper lies in demonstrating that operationally unrestricted spaces (understood as distinct views on the very same geometric or quasigeometric object) makes sense indeed. Hence, the very abstract synthetic demand of mutual compatibility of algebraic/operational and geometric/structural representations of objects can reveal much more than either the purely algebraic approach alone or a purely geometric approach alone could disclose. We should always match algebraic and geometric features on equal footing.

The symbolic prototype formula (19) encapsulates thus the abstract quasigeometric idea of dual reciprocal spatiality whereas the equations (20) – (27) demonstrate how the idea can be algebraically implemented in operationally realistic abstract quasigeometric multispatial hyperspace. The related structural – and in particular the geometric or just quasigeometric – issues shall be further explored and elaborated elsewhere.

9. SOME PHYSICAL IMPORTANCE OF UNRESTRICTED DIVISION BY ZERO

Division by zero seems to be of little importance outside the camp of some algebraists obsessed with purity of possibly unrestricted algebraic operations and some associated with them geometric transformations. However, my main motivation for tackling this algebraic topic sprang from obstacles posed by the infamous prohibition on division by zero to some very difficult problems encountered in physical applications. One example is singularity of gravitational fields. While some singularities have physical consequences, the singularity at zero seems to be a mathematical one. Since both zero and infinity – as natural reciprocal of the real zero – determine the proper domains of validity of many physical theories, we need to explore them comprehensively also in physical context, which shall be done elsewhere.

Schwarzschild singularity is now understood as a peculiarity of the coordinates, for the space near the singularity is perfectly well behaved [144] p. 200. The lesser issue is that of time independence of Schwarzschild metric, which is not postulated but follows from the assumption of spherical symmetry and the presumption of zero stress density [144] p. 156.

Since some apparent singularities are quite inadvertent creations of underdeveloped pure mathematics, we need to explore and fairly well comprehend – at least at the fundamental operational level – the algebraic concepts underlying practical meanings of the most basic abstract mathematical notions that are routinely used also in physical applications. Even the topological approach to issues of string theory requires unrestricted division by zero [3].

10. WHY WAS UNRESTRICTED DIVISION BY ZERO SUPPRESSED?

Why was this quasigeometric division by zero in the quaternionic domain suppressed despite its usefulness for mathematics, physical and astrophysical sciences? Who was so disturbed by the possibility of prospective discovery of some previously unknown realities notwithstanding unbiased experimental evidence of actual existence of the realities?

Prohibition on algebraic division by zero in the real and complex domains presumably emerged from some sort of adversity towards Christ Jesus who is the only one person who has ever actually committed definite and unambiguous division by zero [3].

The adversaries of Christ: satan and his mercenaries – some of whom are often unaware that they virtually serve satan, even if indirectly – were presumably the ones most interested in openly or quietly obfuscating whatever Christ Jesus has ever accomplished or taught. For we wrestle not against flesh and blood, but against principalities, against powers, against the rulers of the darkness of this world, against spiritual wickedness in high places [145].

Why was then the quasigeometric division by zero that can be performed in quaternionic domain – as we have seen – so irritating that it has been tacitly suppressed, if not expressly prohibited? The latter inhibition was most likely instituted to eradicate even thinking of any dividing by zero. For thinking about the possibility of an abstract division by zero might just accidentally ignite an investigation into the feasibility of presence of an extended reality.

Besides the unwarranted SSR paradigm that should be replaced by the MSR paradigm in order to avoid the unrestrained spread of blatant intellectual confusion, all the other dogmas which emerged from the intellectual confusion quite inadvertently, should be rectified too.

Ellen G. White, who was the prophetess of early adventist movement [146] (certified by two medical doctors as truly exhibiting exactly the same behavior as did the ancient prophet Daniel during a vision) asserted that human talent and intelligence count for nothing without relying on the Lord God of heaven [147]. She viewed human brain as aninstrument of nimd [148] just as the software called operating system uses computer device as an instrument for running the system's preprogrammed functions that implement ideas encoded in the system. Since the Bible identified God with truth [149,150], then opposition to any truth implies opposing the God and thus serving as either a virtual or consensual mercenary of satan.

Hence any suppression of truths, whether in sciences or elsewhere, not only impairs the affected sciences but also the suppressor/liar, for the lies create an iniquity (i.e. a perverted state of mind) and thus nontrivial reasonings according to rules of such a corrupted mindset could generate streams of nonsenses posing as valid derivations or as seemingly “logical” deductions. When you refuse light then you have darkness [151]. Recall that even Almighty God cannot lie [152], even though nothing is too hard (i.e. impossible) for God [153].

Catholic church, for example, distorted scriptures and now teaches many falsities such as unconditional immortality of the soul and other nonsenses that contradict the Bible [154] while some other protestant churches follow the nonsensical suit [155]. The Catholic church is identified by God in the book of Revelation of the Bible with the whore of Babylon [156].

Only some Seventh Day Adventist (still defiantly protestant) churches (especially those associated with the AmazingDiscoveries.org still adhere to the truth as it is conveyed by the holy scriptures that were translated from the *textus receptus* of the Bible (such as the King James Bible in English or the Luther’s German translation) generally before AD 1900 [157].

For after the latter date many new or “revised” translations of the Bible became either inadvertently (under the influence of satan) or purposely corrupted [158] just as Jesuits did when they distorted their Douay Bible. One could see the perfidious perversion of Jesuits, which every Jesuit priest does swear and signs by checking their wicked oath [159-161]. Since depravity of Jesuits rivals that of terrorists and murderers and yet nobody is calling them to account for their atrocities, the tacit deception of suppressors of truths testifies how widespread the corruption resulting from perverted mindsets is. Recall that the catholic pope Francis is a Jesuit and that catholic dogmas are firmly grounded in their anti-Christ stance.

Besides the fact that lying or even just suppressing truths hurts others, especially those who stand for the truths, it also hurts some logical reasonings of the suppressor or liar and makes them illogical in general. One just cannot afford to suppress truths and expect to be reasoning correctly on any topics, whether ecclesiastical or scientific. Regardless of what kind of truths are being suppressed, the logical reasoning engine of the suppressor’s mind is adversely affected in all matters where the mind’s engine will be used, and that must mean: elsewhere. By distorting some truths of the word of God as it was given in the scriptures, the Catholic church and her aficionados perpetuate unchallenged nonsenses in all areas of life. The church decreed infallibility of pope in ecclesiastical matters. Yet by altering the God’s laws and thus defying His authority and wisdom, they effectively corrupted their mind’s reasoning engine that cannot even reason properly, not to mention infallibly on any topic where the – broken by corruption – engine is being used. Only their mindless statements could be accidentally true because they do not use their minds’ engines to declare them.

The Catholic church is not the only one that propagates intellectual nonsenses. She is the best example of the widespread intellectual distortions exposed by reformers and lately also by the prophetess Ellen G. White who studied under divine supervision of God Jesus Christ. Her numerous predictions – including the San Francisco 1906 earthquake and the man-made tragedy of Twin Towers in New York in 2001 (without giving any specific dates though) –demonstrate that God is always willing and ready to do for us in supernatural sphere what we think they cannot do in their, thoughtlessly or faithlessly presumed as “natural”, sphere. There is only one reality but several worlds for as Christ Jesus said: My kingdom is not of this world. This is not a call to invoke the dead or satan or his fallen spirits, who are eager to lie and to impersonate the dead, but to repent and turn to God in truth, which means to stop lying. The latter turn implies ceasing to lie to yourself, to others and thus in all sciences too.

The unwarranted traditional prohibition on division by zero, which is perpetuated from kindergarten level to doctoral seminars, cannot be justified by any logical arguments. Yet it is not only tolerated but oftentimes mindlessly defended as well, along the lines of thought also used to support indefensible religious dogmas. I do not argue via similarity of methods, which may be accidental, but by scriptural principles emphasized by Ellen G. White. For as wickedness of people causes deterioration of their natural conditions as well as intellectual abilities of the infected people so also reliance on truth intensifies their intellectual abilities, which is a gift of God. You can read the Bible *ad nauseam* and still not understand it. Only with the help of God the Holy Spirit or a certified prophet one could grasp its significance.

Insofar as mathematics is concerned, lies not only breed nonsenses but could also cause suicidal tendencies in overly sensitive introvert students or violent outbursts against others in students leaning toward the extrovert end of the personality traits spectrum. Since over 1100 students commit suicides each year in the USA alone (not to mention tacitly negated or just overlooked suicidal epidemic in some excessively perfectionist far-eastern Asian and European countries), the issues of faulty but posing as precise mathematics, which is often used as a predictive measure of predisposition for correct thinking, should be reevaluated.

11. CONCLUSIONS

It has been shown that unrestricted algebraic division by zero is quite feasible and can be reasonably implemented via inverse operation of multiplication by infinity, which is treated as the reciprocal of zero also in the domain of quaternionic numbers. The reciprocity of zero and infinity, which is being implemented as reciprocal geometric space within a quasispatial structure of multispatial hyperspace, permits to perform quite unrestricted division not only in the domains of real and complex numbers but also in the abstract realm of 4-dimensional quaternions. There is no mathematical reason to forbid division by zero in these domains. Since 4D quaternions comprise both vectors and scalars, the quite unrestricted operation of division – including abstract division by zero – prompted the introduction of certain new differential operators suitable for operating on pairs of abstract geometric or quasigeometric dual reciprocal spaces composing the hyperspace.

Quaternionic division by zero reveals presence of few (unanticipated mathematically in the past) features of the abstract mathematical structure of the physical reality we live in.

Since the traditional mathematics clearly contradicts itself not only on some operational issues but also on abstract structural conceptions, and is challenged even by some unbiased experimental evidence, it is predictably untenable in its traditional form.

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