# Motion of charged particles in Minkowski spacetime

**J. H. Caltenco[1], J. López-Bonilla[1,*], Laurian Ioan Piscoran[2]**

[1]ESIME-Zacatenco, Instituto Politécnico Nacional, Col. Lindavista CP 07738, CDMX, México

[2]Computer Sci. North Univ. Center of Baia Mare, Victoriei 76, 430 122 Baia Mare, Romania

*E-mail address: jlopezb@ipn.mx

**ABSTRACT**

The electromagnetic field generated by a charge in arbitrary motion in Minkowski space is briefly studied. Particularly important is the deduction of superpotential for the radiative part of the Maxwell tensor.

*Keywords*: Liénard-Wiechert field, Faraday tensor, Minkowski spacetime, Superpotentials for the bounded and radiative parts of Maxwell tensor

## 1. INTRODUCTION

A charge in arbitrary motion in special relativity generates the Liénard-Wiechert retarded potential $A_r$ and its corresponding Faraday tensor $F_{rs}$ which is of fundamental importance in point particle electrodynamics. Accordingly, we shall dedicate Sec. 1 to the study of scalar and vectorial quantities associated to the world line of the charge, with special emphasis on retarded distance and the light cone: the trajectory's kinematics forms a powerful platform for analysis of the electromagnetic field. Additionally, the valuable Fermi's triad is introduced.

In Sec. 2 we consider general aspects regarding 4-potential and Faraday tensor, bringing them into Synge classification [1] and an attractive theorem of Stachel [2]. Sec. 3 concerns the Liénard-Wiechert case, obtaining the Teitelboim [3, 4]-Miglietta [5] decompositions of $A_r$ and $F_{cb}$, respectively. We also deduce Plebañski's interesting result [6]: $F_{ij}$ is the antisymmetric

product of two gradients. Sec. 4 deals with retarded Maxwell tensor, its study is channeled through the algebraic-differential properties of their $T_{ic}^{B}$ and radiative $T_{ab}^{R}$ parts. A thoughtful analysis of Weert superpotential structure is made for - $T_{rs}^{B}$ , and the non-local superpotential is shown for - $T_{ij}^{R}$ . In our whole study, the Synge article [7] is fundamental for the mathematical aspects of point particles electrodynamics.

## 2. KINEMATICS OF RELATIVISTIC PARTICLES

In special relativity, a "particle" means a timelike world line, see Fig. 1, whose unitary tangent vector is the 4-velocity:

$$v^r = dx^r / d\tau, \tag{1.a}$$

where the proper time $\tau$ is defined by:

$$d\tau^2 = g_{ab}dx^a dx^b = -dx^2 - dy^2 - dz^2 + dt^2, \tag{1.b}$$

which means that the metric is *Diag* (-1, -1, -1, 1) and $c$ = light's speed in vacuum = 1, then:

$$\left(v^r\right) = \left(\gamma\vec{v}, \gamma\right) \quad \text{with} \quad \vec{v} = \left(dx/dt, dy/dt, dz/dt\right) \quad \text{and} \quad \gamma = \left(1 - \vec{v}^2\right)^{\frac{1}{2}}. \tag{1.c}$$

So, out of (1.a, b) we have that:

$$v^r v_r = -1, \qquad v^r a_r = 0 \quad \text{with} \quad a^r = dv^r / d\tau = \text{4-acceleration}, \tag{1.d}$$

this implies the time-like and space-like nature of $v^r$ and $a^r$, respectively; in consequence:

$$a^2 \equiv a_r a^r \geq 0; \tag{1.e}$$

from (1.d) we obtain:

$$s^r v_r + a^2 = 0, \qquad \text{where} \qquad s^r = da^r / d\tau = \text{superacceleration}. \tag{1.f}$$

In Fig. 1, we have not indicated this last vector since it might be outside or inside the light cone.
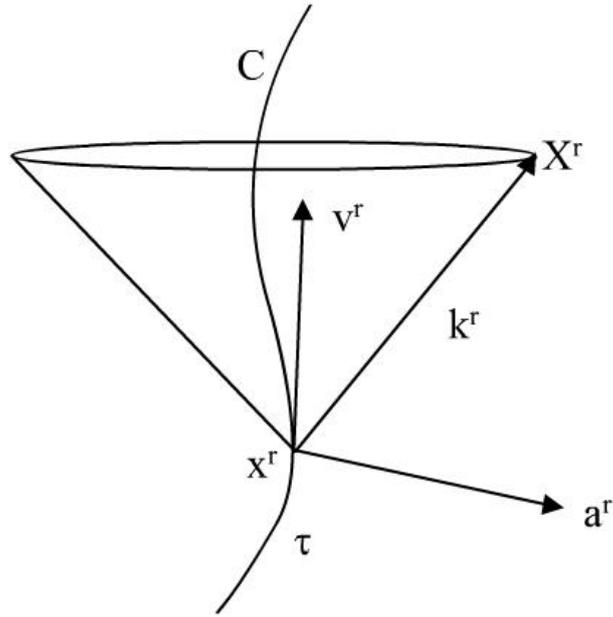
**Figure 1.** Time-like trajectory

From an event $X^r$ outside $C$ we trace its null cone's past sheet which intersects to $C$ in the point $x^r$ called "retarded event associated with $X^r$", so we say that:

$$x^r = x^r\left(X^i\right) \tag{2.a}$$

thus with $X^i$ given, the retarded point over $C$ is automatically determined. This allows us to introduce the vector:

$$k^r = X^r - x^r, \tag{2.b}$$

of magnitude zero, because it rests over the cone:

$$k^r k_r = 0, \tag{2.c}$$

so, $k^r$ indicates the propagation direction of the photons emitted by the particle. The null or light type vector (2.b) is truly important in electrodynamics: we could say that studying the Maxwell field is almost equivalent to an analysis of the null cone and its relation to the world line. From $X^r$ we can build two distances widely used in the study of charges in Minkowski space:
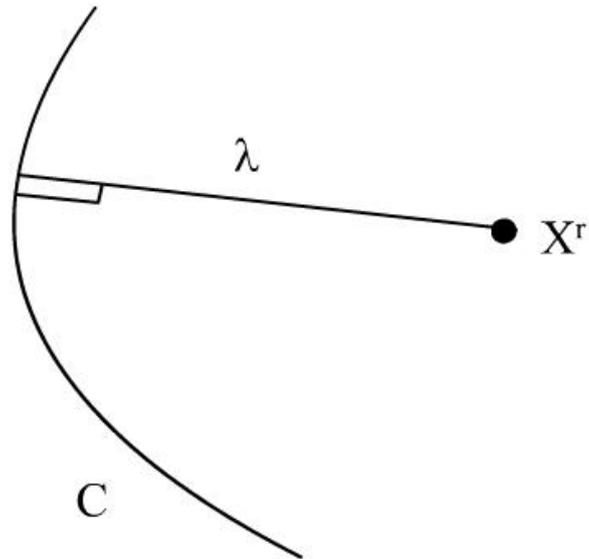
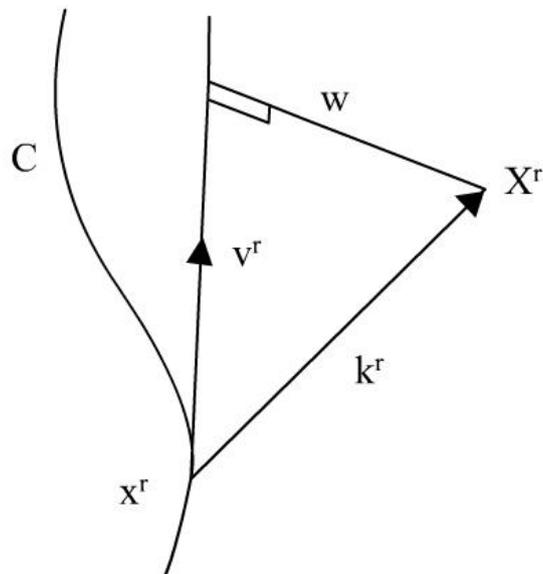**Figure 2a.** Instantaneous distance from $X^r$ to $C$.



**Figure 2b.** Retarded distance from $X^r$ to $C$.

The instantaneous distance (see Fig. 2.a) introduced by Dirac [8] is geometrically simpler than retarded distance $w$, (see Fig. 2.b) proposed by Bhabha [9] and furthered by Synge [7]; nevertheless, $\lambda$ has the big disadvantage of not involving retarded effects (light cone); for this reason, $w$ has more physical meaning and leads to simpler calculations because

it intrinsically takes in account the finite velocity of interaction. Here we will work only with *w*, whose expression is given by:

$$w = -k^r v_r \geq 0. \tag{3.a}$$

Bringing to mind that a null vector cannot be orthogonal to a time-like one, (3.a) points out that:

$$w = 0 \quad \text{if and only if} \quad k^r = 0, \tag{3.b}$$

in other words, the retarded distance is zero only when $X^r$ is over *C*.

When making calculations, we need to know how diverse quantities change over *C* when an external event $X^r$ varies; for this, it is enough with having change's law for $\tau$ because $x^r$, $v^r$, $a^r$, etc. are functions of this parameter:

$$\tau_{,r} = -w^{-1} k_r \quad \text{where} \quad ,r = \partial / \partial X^r, \tag{4}$$

so we have that $\tau_{,r}$ is a null vector because it is anti-parallel to $k_r$. Every event $X^r$ over the same cone has an associated unique value of $\tau$, that is, the light cone is the $\tau$ = constant surface, so $\tau^r$ is the vector normal to the cone even though our Euclidian eyes don't see it like that. Due to (4) it makes no sense to look for a unitary normal to the cone. Thanks to (4) it is easy to obtain these useful relationships:

$$x^r_{,j} = -w^{-1} v^r k_j, \quad v^r_{,j} = -w^{-1} a^r k_l, \quad a^r_{,b} = -w^{-1} s^r k_b, \quad k^r_{,C} = \delta^r_C + w^{-1} v^r k_C,$$

$$w_{,C} = -v_C + B k_c, \quad B = w^{-1}(1-W), \quad W = -k^r a_r, \quad W_{,b} = W_b = -a_b + w^{-1} k^r s_r k_b,$$

$$B_{,C} = w^{-1} \left[ U_C - \left( B^2 + w^{-1} k^r s_r \right) k_c \right], U_C = B v_C + a_C, U^C k_c = -1 \quad U^C v_C = -B, \tag{5}$$

$$U^C a_C = a^2, \quad U^C U_C = a^2 - B^2, \quad U^c w_{,c} = 0, \quad U^C_{,C} = 0.$$

In relativity, a spatial triad of vectors is also important at each point of the curve because this triad is a local frame of reference for an observer mounted on the particle, see Fig. 3:

$$\left( e_{(a)}{}^r e_{(b)r} \right) = Diag(1,1,1,-1); \tag{6.a}$$

this tetrad forms a base for each vector associated to the world line, in particular for null vector (2.b):

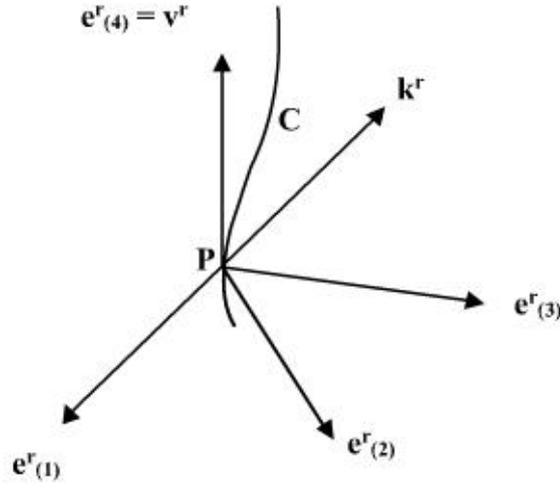$$k^r = b^\sigma e_{(\sigma)}{}^r + b^4 e_{(4)}{}^r. \tag{6.b}$$

**Figure 3.** Orthonormal tetrad.

From now on the Greek indexes shall only take values 1, 2, and 3. Expansion (6.b) can be written as follows:

$$k^r = M^r + b^4 v^r \quad \text{with} \quad M^r = b^\sigma e_{(\sigma)}{}^r, \qquad M^r v_r = 0, \tag{6.c}$$

$M^r$ is space-like type because it is a lineal combination of the three space-like vectors of the tetrad, see Fig. 4. If $M = \left(M^r M_r\right)^{\frac{1}{2}}$ is the magnitude of $M^r$, then:

$$M^r = M p^r \quad \text{with} \quad p^r p_r = 1, \tag{6.d}$$

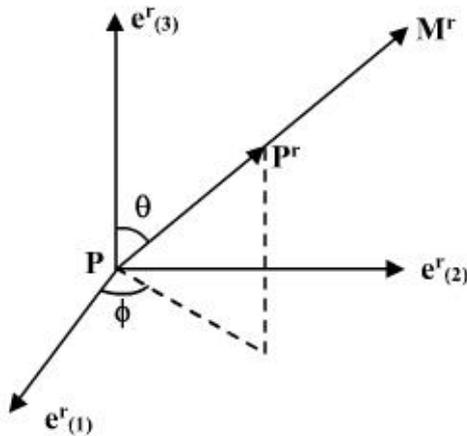and by (6.c):

$$p_r v^r = 0, \tag{6.e}$$



**Figure 4.** Spatial triad

so $p^r$ is a space-like unitary vector. From (2.c, 3.a) it is plain that $M = b^4 = w$, and as a consequence (6.b,…,e) implies the important Synge [7] - Teitelboim [4] decomposition for $k^r$:

$$k^r = w\left(p^r + v^r\right), \quad p^r v_r = 0 \quad \text{and} \quad p_r k^r = w, \tag{7}$$

which is shown in the following two figures:
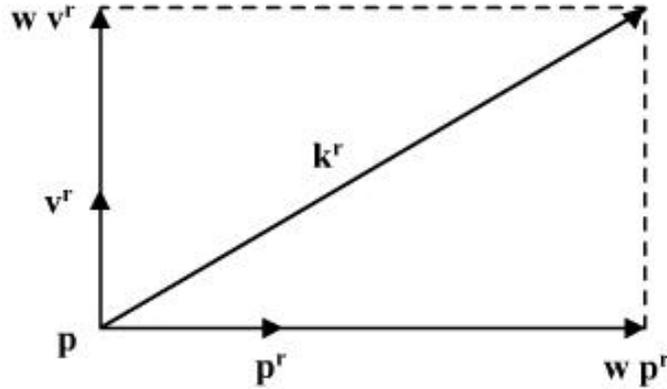


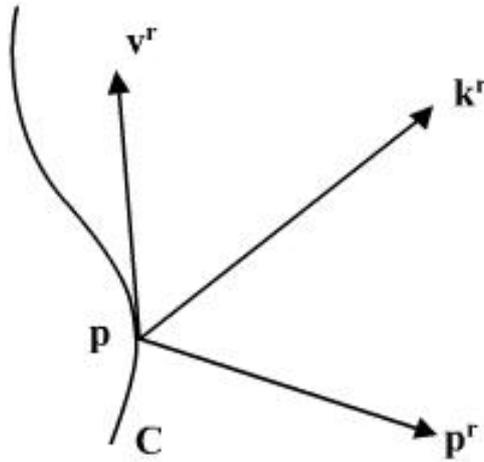**Figure 5.a.** Spatial and time-like components of $k^r$.



**Figure 5.b.** Null vector $k^r$ splitting

The unitary vector $p^r$ only depends on the spatial triad; so it can be written with common spherical coordinates (see Fig. 4):

$$p^r = \sin\theta\cos\phi e_{(1)}{}^r + \sin\theta\sin\phi e_{(2)}{}^r + \cos\theta e_{(3)}{}^r = p^{(\sigma)}e_{(\sigma)}{}^r \tag{8.a}$$

$$\therefore \quad p^{(\gamma)} = e^{(\gamma)}{}_r p^r \,,$$

where we have employed the dual base $e_{(\square)}{}^r$, defined by:

$$e^{(\gamma)r} e_{(\sigma)r} = \delta^\gamma_\sigma \,; \tag{8.b}$$

vector $p^r$ does not necessarily have to be orthogonal to 4-acceleration $a^r$.

The triad $e_{(\gamma)}{}^r$ is arbitrary except for the orthonormality conditions (6.a); nevertheless, some triads may be more convenient than others in some set calculations. For our theoretical purposes, the Fermi triad [10] is very important; it satisfies over $C$ the transport law (which we use in this work):

$$de_{(\sigma)}{}^r / d\tau = e_{(\sigma)}{}^b a_b v^r = a_{(\sigma)} v^r \,. \tag{9.a}$$

This type of transport has been very fundamental in gravitation, for example, Pirani [11] and Synge [12]; but, in electrodynamics we shall show its participation in the deduction of superpotential for the radiative part of the Maxwell tensor, see Sec. 4. In (9.a) we have used the notation:

$$a_{(\sigma)} = a^r e_{(\sigma)r} \,, \tag{9.b}$$

because $a^r$ is space-like type; remember that the triad is only defined over $C$.

To end this Section, we give some useful expressions:

$$w_{,b} k^b = w \,, \qquad w_{,b} v^b = W \,, \qquad w_{,b} a^b = -WB \,, \qquad w^b w_{,b} = 1 - 2W \,,$$

$$W = -w p^r a_r = -w p^{(\sigma)} a_{(\sigma)} \,, \qquad W_{,c} k^c = W \,, \qquad W_{,c} w^c = WB + k^r s_r \,, \qquad k^r{}_{,a} v^a = 0 \,,$$

$$B_{,c} k^c = -w^{-1} \,, \quad w_{,c} p^c = wB \,, \quad w^{-1,a}{}_{,a} = 0 \,, \quad w^a{}_{,a} = 2w^{-1}(1 - 2W) \,, \quad k^r{}_{,a} p_r = p_a \,, \tag{10}$$

$$U^r p_r = -w^{-1} W \,, \quad p^r{}_{,a} = w^{-1} \left[ \delta^r_a + w^{-1} v_a k^r + \left( a^r + w^{-1} v^r - w^{-1} B k^r \right) k_a \right] \,, \quad p^r{}_{,a} k^r = -w^{-1} W k_a$$

$$p^r{}_{,a} w_{,r} = w^{-2} W^2 k_a \,, \quad p_{(\sigma),r} k^r = 0 \,, \quad p^r{}_{,a} k^a = 0 \,, \quad p^r{}_{,r} = w^{-1}(2 - W) \,, \quad p_{(\sigma),r} = p_{a,r} e_{(\sigma)}{}^a$$

Relations (5, 10) are the basic formulary for any calculation in the electrodynamics of classical charged particles.

## 3. 4-POTENTIAL AND FARADAY TENSOR

In this Section we consider the algebraical and differential properties satisfied by the electromagnetic field in vacuum. *The* Faraday tensor is given by:

$$F_{rc} = -F_{cr} = A_{c,r} - A_{r,c},$$ (11.a)

in terms of 4-potential $A^b$; from (11.a) the fulfillment of the cyclic relationship is clear:

$$F_{br,c} + F_{rc,b} + F_{cb,r} = 0, \quad \text{in other words} \quad {}^*F^{rc}{}_{,c} = 0,$$ (11.b)

where we have employed the dual tensor:

$${}^*F^{rc} = -{}^*F^{cr} = \frac{1}{2}\varepsilon^{rcab}F_{ab},$$ (11.c)

such that $\varepsilon^{ijkr}$ is the Levi-Civita antisymmetric symbol. In free space we have the remaining Maxwell equation:

$$F^{rc}{}_{,c} = 0,$$ (11.d)

which in turn leads to a differential equation for 4-potential:

$$A^{r,c}{}_{,c} - \left(A^c{}_{,c}\right)^{,r} = 0.$$ (11.e)

In (11.a) we have full freedom to add an arbitrary gradient to $A^r$ without modifying the Faraday tensor, then without lack of generality we can always demand:

$$A^c{}_{,c} = 0 \qquad \text{Lorenz-Riemann condition;}$$ (11.f)

simplifying (11.e):

$$A^{r,c}{}_{,c} = 0 \qquad \text{Wave equation.}$$ (11.g)

So, from the mathematical point of view, the problem consists of solving (11.g) with the restriction (11.f), which matches solving (11.b,d); in other words:

$$\nabla \cdot \vec{B} = 0 , \quad \nabla \times \vec{E} = -\partial \vec{B}/\partial t , \quad \nabla \cdot \vec{E} = 0 \quad \text{and} \quad \nabla \times \vec{B} = \partial \vec{E}/\partial t ,$$ (12)

in the MKS system; remembering that $c = \left(\varepsilon_0 \mu_0\right)^{-1/2} = 1$.

The Faraday matrix representation becomes:

$$[F^{ab}] = \begin{bmatrix} 0 & B_z & -B_y & -E_x \\ -B_z & 0 & B_x & -E_y \\ B_y & -B_x & 0 & -E_z \\ E_x & E_y & E_z & 0 \end{bmatrix} \begin{matrix} a:rows \\ \\ \\ b:columns \end{matrix}, \tag{13.a}$$

and with (11.c) an associated matrix for the dual tensor can be constructed:

$$[^*F^{ab}] = \begin{bmatrix} 0 & E_z & -E_y & B_x \\ -E_z & 0 & E_x & B_y \\ E_y & -E_x & 0 & B_z \\ -B_x & -B_y & -B_z & 0 \end{bmatrix}. \tag{13.b}$$

Note that (13.b) is obtained if we make the following changes to (13.a):

$$\vec{B} \to \vec{E} \quad, \qquad \vec{E} \to -\vec{B}; \tag{13.c}$$

So it may come to mind that * executes operation (13.c), then it is clear that $^{**}F^{ar} = -F^{ar}$.

Comparing (11.a, 13.a) we obtain the relationship of the electric and magnetic fields with 4-potential:

$$\vec{E} = -\nabla\phi - \partial\vec{A}/\partial t \quad, \qquad \vec{B} = \nabla\times\vec{A}, \tag{14}$$

because $(A^r) = (\vec{A}, \phi)$, where $\vec{A}$ and $\phi$ are the magnetic and electric potentials, respectively. We are placing emphasis on $\phi$ as a scalar, but not as an invariant; the electromagnetic field only possesses two Lorentz invariants, namely:

$$F_1 \equiv F_{ab}F^{ab} = 2(\tilde{B}^2 - E^2), \qquad F_2 \equiv {}^*F_{ab}F^{ab} = 4\vec{E}\cdot\vec{B}, \tag{15}$$

with $E = |\vec{E}|$ and $\tilde{B} = |\vec{B}|$.

Just like Weyl tensor invariants allow to establish the Petrov classification [13] for the gravitational field, the quantities (15) lead to the Synge [1] – Piña [14] classification for the Faraday tensor:

$$\text{Type A:} \qquad F_2 \neq 0$$

$$\text{Type B:} \qquad F_1 < 0 \quad \text{and} \quad F_2 = 0$$

$$\text{Type C:} \qquad F_1 = 0 \quad \text{and} \quad F_2 = 0 \qquad \text{Null field} \tag{16}$$

$$\text{Type D:} \qquad F_1 > 0 \quad \text{and} \quad F_2 = 0$$

A point with a null field means that, in such an event, $\vec{E} \perp \vec{B}$ and $E = \tilde{B}$. A non-null field implies a different type of C. Classification (16) is algebraic but the type of electromagnetic field may change from one point to another. Furthermore, we will see that the field that produces a relativistic charge is B type, which tends to type C (plane wave) towards infinity.

There are very important identities for the Maxwell field:

$$F^{ar}F_{br} - {}^{*}F^{ar}\,{}^{*}F_{br} = \left(F_1/2\right)\delta_b^a\,, \tag{17.a}$$

$$ {}^{*}F^{ar}F_{br} = \left(F_2/4\right)\delta_b^a\,, \tag{17.b}$$

which do not have a specific name, are well known and can be found in Rainich [15], Plebañski [16], Wheeler [17] pp. 239, Penney [18] and Piña [14]. Expressions (17) correspond to Lanczos identities [18] between the Riemann tensor and its double dual. If (17.a) is multiplied by $F_{ia}$ or ${}^{*}F_{ia}$ and (17.b) is employed, valuable identities result in the calculation of an antisymmetric matrix's exponential function [14, 20]:

$$F_{ia}F^{ar}F_{rb} = \left(F_1/2\right)F_{bi} + \left(F_2/2\right){}^{*}F_{bi}\,,$$

$$ {}^{*}F_{ia}\,{}^{*}F^{ar}\,{}^{*}F_{rb} = \left(F_2/4\right)F_{bi} - \left(F_1/4\right){}^{*}F_{bi}\,. \tag{18}$$

From (13.a) it is simple to show that:

$$\det\left(F^{ab}\right) = \left(1/16\right)\left(F_2\right)^2 = \left(\vec{E}\cdot\vec{B}\right)^2\,, \tag{19.a}$$

which is a particular case of the following theorem, see Drazin [21]:

"The expression $\det \underset{\sim}{F}$, with antisymmetric $\underset{\sim}{F}_{n\times n}$ and even $n$, is the square

of a rational polynomial in $F^{ij}$" $\tag{19.b}$

Now we mention the interesting and useful Stachel theorem [2]:
"If $\underset{\sim}{F}$ satisfies:

$$F_{ar} = -F_{ra}\,, \quad F_{ar,c} + F_{rc,a} + F_{ca,r} = 0\,, \qquad \det\left(F_{ab}\right) = 0\,, \tag{20.a}$$

then there exist functions $\beta$ and $\psi$ such that:

$$F_{ar} = \beta_{,a}\psi_{,r} - \beta_{,r}\psi_{,a} \text{ ". } \tag{20.b}$$

That is, the conditions (20.a) reduce $\underset{\sim}{F}$ to an antisymmetric product of gradients. If we extend the Stachel result to the Maxwell field, then the first two conditions (20.a) will be immediately verified, thereby:

"If the Faraday tensor fulfills $F_2 = 0$, then it has the form (20.b)".         (20.c)

The Liénard-Wiechert solution satisfies (20.c), and thus permits us to write $\underset{\sim}{F}$ in the form of Plebañski [6]. In general, an electromagnetic field with a different type of A has the structure (20.b). Results (20) are valid in the presence of curvature because its differential expressions remain undisturbed if covariant derivatives are used instead of partial ones:

$$^{*}F^{ar}_{,r} = {}^{*}F^{ar}_{;r} = 0 \quad , \quad \beta_{,r} = \beta_{;r}$$

In the following Section we shall employ the material explained in (11 - 20) to analyze the field produced by a point charge with a relativistic trajectory.

## 4. THE LIÉNARD-WIECHERT FIELD

The solution of (11.f, g) for a particle in arbitrary motion in Minkowski space was obtained by Liénard and Wiechert; the corresponding potential carries their names and is given by:

$$A^r\left(X^b\right) = qw^{-1}v^r \ , \quad q = \text{charge}/4\pi\varepsilon_0 \quad \text{Retarded potential,} \qquad (21.a)$$

which is fundamental in everything that follows; by the use of (11.a) it is simple to calculate the associated Faraday tensor :

$$F_{rb} = qw^{-2}\left(U_r k_b - U_b k_r\right) = qw^{-2}U_r \times k_b \qquad (21.b)$$

where $\times$ means the antisymmetric product. This notation is due to Lowry [22] and will cause such expressions to be very compact. From (11.c, 21.b) it is clear that:

$$^{*}F_{mn} = -qw^{-2}\varepsilon_{mnab}U^a k^b , \qquad (21.c)$$

therefore:

$$F_1 = -2q^2 w^{-4} < 0 \quad , \quad F_2 = 0 . \qquad (21.d)$$

In other words, the electric and magnetic fields generated by the charge satisfy:

$$\tilde{B} < E , \qquad \vec{E}\cdot\vec{B} = 0 , \qquad (21.e)$$

in consequence $\underset{\sim}{F}$ is type B. Note that in the asymptotic region ($w \to \infty$), the invariant $F_1$ tends to zero, which means that $\underset{\sim}{F}$ is close to the null case (type C) far away from the charge.

With (21.d), (20.c) is valid, so the Stachel theorem [2] implies that (21.b) can be reduced to (20.b). This is easy to do because from (4, 5) the following relationships are available:

$$k_r = -w\tau_{,r} \ , \qquad U_r = wB_{,r} + \left(B^2 + w^{-1}k^c s_c\right)k_r \ ,$$

which substituted in (21.b) implies that:

$$F_{rc} = -qB_{,r}x\tau_{,c} = q\left(\tau_{,r}B_{,c} - \tau_{,c}B_{,r}\right), \tag{22}$$

with the form (21.b), meaning that $\tau$ and $B$ correspond to functions $\beta$ and $\psi$; expression (22) was first obtained by Plebañski [6].

Now we shall consider the eigenvalue problem of $\underset{\sim}{F}$; for this purpose we stem from (21.b), and due to:

$$U_r k^r = -1 \qquad , \qquad k_r k^r = 0 \ ,$$

then we immediately have one of the two null proper vectors of a non-null field (different type from C) [23]:

$$F_{rm}k^m = qw^{-2}k_r, \qquad \text{proper value} = qw^{-2}, \tag{23.a}$$

this suggests that $U_r$ may be an eigenvector, but it isn't:

$$F_{rb}U^b = -qw^{-2}U_r - qw^{-2}\left(a^2 - B^2\right)k_r. \tag{23.b}$$

Nevertheless, if we multiply (23.a) by $\frac{1}{2}\left(a^2 - B^2\right)$ and add the resulting equation to (23.b), we obtain the other null proper vector [24, 25]:

$$F_{rm}\eta^m = -qw^{-2}\eta_r \qquad , \qquad \text{proper value} = -qw^{-2},$$

with:

$$\eta^r = U^r + \frac{1}{2}\left(a^2 - B^2\right)k^r \qquad , \qquad \eta^r \eta_r = 0 \ . \tag{23.c}$$

It is not usual to find $\eta^r$ explicitly in the literature; it is clear that these two proper vectors are independent because:

$$k^r \eta_r = 1 \ . \tag{23.d}$$

Remember that two null vectors $\xi^r$ and $\gamma^r$ are proportional if and only if $\xi^r \gamma_r = 0$, so (23.d) implies the non-parallelism of such proper vectors.

With (10, 21.b) it is possible to prove that:

$$F_r^b p_{c,b} = qw^{-4}Wv_c k_r , \qquad F_r^b p_{(\sigma),b} = 0 , \tag{24}$$

of great importance in the next Section on the deduction of the radiative superpotential

Teitelboim started an era in electrodynamics by employing only retarded fields and studying Faraday and Maxwell tensors near to and away from a point charge. This type of analysis is generated by substituting (5, 7) in (21.b) to obtain the following decomposition:

$$F_{rb} = \underset{(-1)^{rb}}{F} + \underset{(-2)^{rb}}{F} , \tag{25.a}$$

where:

$$\underset{(-1)^{rb}}{F} = qw^{-2}\left(w^{-1}Wv_r + a_r\right) \times k_b , \tag{25.b}$$

$$= qw^{-1}\left(a_c p^c v_r \times p_b + a_r \times v_b + a_r \times p_b\right), \tag{25.c}$$

$$\underset{(-2)^{rb}}{F} = qw^{-3}v_r \times k_b = qw^{-2}\left(v_r \times p_b\right) . \tag{25.d}$$

Meaning that $\underset{(-i)^{rb}}{F}, i = 1,2$ varies like $w^{-i}$; the dependence on $w^{-i}$ is clear because the parenthesis in (25.c, d) are independent of the retarded distance; their terms are functions of $x^r$ which remains stationary when we move away over the light cone. Thus $\underset{(-1)^{rb}}{F}$ and $\underset{(-2)^{rb}}{F}$ are dominant away from ($w>>1$) and near to ($w<<1$), respectively, then $\underset{(-1)^{ij}}{F}$ being responsible for the Larmor formula which provides the radiation speed towards infinitum.

Note that (25.b, c) depends on the particle acceleration, which is an expected result because of the Schild theorem [26]:

$$\text{"Radiation exists if and only if } a^r \neq 0 \text{".} \tag{26}$$

Schild was the first author to give a covariant definition of radiation even though some of his ideas were already implicit in Synge [27], Appendix A, whose 1st edition was made in 1955. We call $\underset{(-1)^{ij}}{F}$ the radiative part of $F_{ab}$ because it is a null field in classification (16):

$$\underset{(-1)^{ra}}{F}\underset{(-1)}{F}^{ra} = \overset{*}{\underset{(-1)^{ra}}{F}}\underset{(-1)}{F}^{ra} = 0, \tag{27.a}$$

this doesn't happens with $\underset{(-2)^{ij}}{F}$:

$$\underset{(-2)^{ra}}{F}\ \underset{(-2)}{F^{ra}} = 2q^2 w^{-4} < 0, \qquad \underset{(-2)^{ra}}{^*F}\ \underset{(-2)}{F^{ra}} = 0, \tag{27.b}$$

which belongs to type B, this portion will be designated as the bounded part of $F_{mni}$, besides:

$$\underset{(-1)^{ab}}{F}\ \underset{(-2)}{F^{ab}} = 0, \qquad \underset{(-1)^{ab}}{^*F}\ \underset{(-2)}{F^{ab}} = \underset{(-1)^{ab}}{F}\ \underset{(-2)}{^*F^{ab}} = 0; \tag{27.c}$$

the relations (27) can be found in Weert [28].

It is possible to write (25.a) in the form:

$$F_{ab} = w^{-1} N_{ab} + w^{-2} M_{ab}, \qquad \text{so that} \qquad N_{ab} = w \underset{(-1)^{ab}}{F}, \qquad M_{ab} = w^2 \underset{(-2)^{ab}}{F}, \tag{28}$$

with the following properties:

$$N_{ab}\xi^b = 0, \qquad M_{ab}\xi^b = q\xi_a, \qquad \xi_r\xi^r = 0, \qquad \xi_r = w^{-1}k_r = -\tau_{,r}.$$

So we see that (28) is coherent with (1.1, 2, 3) of the Goldberg-Kerr theorem [29] for the asymptotic behavior of electromagnetic fields.

Teitelboim's decomposition (25.a) is fundamental in everything that follows, and it is interesting that (7) generates such splitting in a natural manner:

$$v^r = w^{-1}k^r - p^r,$$

which substituting in (21.a) gives:

$$A^r = \underset{1}{A^r} + \underset{2}{A^r} \qquad \text{with} \qquad \underset{1}{A^r} = -qw^{-1}p^r, \qquad \underset{2}{A^r} = qw^{-2}k^r. \tag{29.a}$$

This partition of the Liénard-Wiechert 4-potential is found in the Teitelboim [4] well-known article; however, it was also published by Migglietta [5] not-knowing Ref. [4]. Expressions (29.b) are simpler than Migglietta's (2.3, 3.2). Placing (29.a) into (11.a) we obtain the matching Faraday's tensor decomposition (25.a) with:

$$\underset{(-i)^{bc}}{F} = \underset{i}{A_{c,b}} - \underset{i}{A_{b,c}} \quad , \quad i = 1, 2, \tag{29.b}$$

which means that each part of $F_{mn}$ has its own 4-potential. At last, it can be verified that (29.a) does not satisfy the Lorenz-Riemann condition (11.f):

$$\underset{1}{A^r_{,r}} = -\underset{2}{A^r_{,r}} = -qw^{-2} \neq 0. \tag{29.c}$$

## 5. ENERGY-MOMENTUM TENSOR

Now we shall consider the Maxwell tensor $T_{rb}$ through which an electromagnetic field's content of energy-momentum is quantified:

$$T_{ab} = \tfrac{1}{2}\left( F_{ac}F_b^c + {}^*F_{ac}\,{}^*F_b^c \right) \;,\tag{30.a}$$

this satisfies:

$$T_{ab} = T_{ba} \qquad \text{Symmetry,} \tag{30.b}$$

$$T_r^r = 0 \qquad \text{Null trace,} \tag{30.c}$$

$$T_{ac}T_b^c = \tfrac{1}{4}\left( T_{mn}T^{mn} \right) g_{ab} \qquad \text{Rainich identity.} \tag{30.d}$$

Symmetry (30.b) is a property of every energy tensor, (30.c) tells us that the field is made of particles with null mass at rest, photons in this case; (30.d) was obtained by Rainich [15].

If we employ (17.a) in the second term of (30.a) we obtain an alternative expression for the Maxwell tensor:

$$T_{ab} = F_{ac}F_b^c - \left( F_1/4 \right) g_{ab}; \tag{30.e}$$

substitution of (21.d, 25) in (30.e) results in the important Teitelboim splitting [3]:

$$T_{ab} = \underset{(-2)^{ab}}{T} + \underset{(-3)^{ab}}{T} + \underset{(-4)^{ab}}{T} \;,\tag{31.a}$$

where:

$$\underset{(-2)^{rn}}{T} = \underset{(-1)^{rc}}{F}\,\underset{(-1)^{n}}{F^c} = q^2 w^{-4}\left( a^2 - w^{-2}W^2 \right) k_r k_n \;,\tag{31.b}$$

$$\underset{(-3)^{rn}}{T} = \underset{(-1)^{rc}}{F}\,\underset{(-2)^{n}}{F^c} + \underset{(-1)^{nc}}{F}\,\underset{(-2)^{r}}{F^c} = q^2 w^{-4}\left[ k_r a_n + k_n a_r + 2w^{-2}W k_r k_n - w^{-1}W\left( k_r v_n + k_n v_r \right) \right],\tag{31.c}$$

$$\underset{(-4)^{rn}}{T} = \underset{(-2)^{rc}}{F}\,\underset{(-2)^{n}}{F^c} - \left( F_1/4 \right) g_{rn} = q^2 w^{-4}\left[ \tfrac{1}{2} g_{rn} + w^{-1}\left( v_r k_n + v_n k_r \right) - w^{-2}k_r k_n \right],\tag{31.d}$$

with the following properties:

$$\underset{(-2)^{rn}}{T}\,k^n = 0, \qquad \underset{(-3)^{rn}}{T}\,k^n = 0, \qquad \underset{(-4)^{rn}}{T}\,k^n = -\left( q^2/2 \right) w^{-4} k_r \;.\tag{31.e}$$

From (31.a, e) it is clear that $k^r$ is a null proper vector of the Maxwell tensor:

$$T_{rn}k^n = -\left(q^2/2\right)w^{-4}k_r , \tag{32}$$

which was to be expected due to (21.d, 23.a, 30.e):

$$T_{rn}k^n = -qw^{-2}F_{rn}k^n + \left(q^2/2\right)w^{-4}k_r = -\tfrac{1}{2}q^2 w^{-4}k_r ,$$

identical to (32). The notation $\underset{(-i)^{rn}}{T}, i=2,3,4$ evokes that (31.b, c, d) vary like $w^{-i}$, in consequence:

$$\underset{(-2)^{ab}}{T} \qquad \text{Dominates when } w\to\infty \text{ (away from the charge),}$$

$$\tag{33}$$

$$\underset{(-3)^{ab}}{T} \,\&\, \underset{(-4)^{ab}}{T} \qquad \text{Dominates when } w\to 0 \text{ (close from q),}$$

so the Larmor formula comes from $\underset{(-2)^{ab}}{T}$. It and $\underset{(-i)^{ab}}{T}, i=3,4$ are responsible for the singularities in the point charge's position, so Teitelboim wrote (31.a) in the form:

$$T_{rn} = \underset{R^{rn}}{T} + \underset{B^{rn}}{T} , \tag{34.a}$$

where:

$$\underset{R^{rn}}{T} = \text{Radiative part} = \underset{(-2)^{rn}}{T} = q^2 w^{-4}\left(a^2 - w^{-2}W^2\right)k_r k_n , $$

$$\tag{34.b}$$

$$\underset{B^{rn}}{T} = \text{bounded part} = \underset{(-3)^{rn}}{T} + \underset{(-4)^{rn}}{T}$$

$$= q^2 w^{-4}\left[\tfrac{1}{2}g_{rn} + \left(k_r a_n + k_n a_r\right) + B\left(k_r v_n + k_n v_r\right)w^{-2}\left(1-2W\right)k_r k_n\right]$$

this proves that such parts are dynamically independent, which means that they verify separately (outside the world line):

$$\underset{R^{rn}}{T^n}_{,n} = 0 , \tag{35.a}$$

$$\underset{B^{rn}}{T^n}_{,n} = 0 . \tag{35.b}$$

It is simple to obtain the relations:

$$T_{rn}U^n = \lambda U_r \qquad T_{rn}B^n = \lambda B_{,r} \qquad T_{rn}\eta^n = \lambda \eta_r , \qquad \lambda = -\left(q^2/2\right)w^{-4}. \tag{36}$$

So we have that $F_{aj}$ and $T_{rc}$ have the same null proper vectors, which is a general result (see Synge [27], p. 337); Plebañski [6], p. 41, was the first one to observe that $B_r$ is a proper vector of $T_{ij}$. If we substitute (34.b, c) in (34.a) we obtain the Synge [7] compact expression for the energy tensor associated to the Liénard-Wiechert retarded potential:

$$T_{rn} = q^2 w^{-4} \left[ k_r U_n + k_n U_r + \left(a^2 - B^2\right) k_r k_n + \tfrac{1}{2} g_{rn} \right]. \tag{37}$$

Weert's [30, 31] attention was driven forward to the fact that (35) are valid *identically,* and he therefore suggested the existence of superpotentials for the bounded and radiative parts. However, he only obtained successfully the explicit form of the superpotential (which now carries his name) $\underset{B^{sar}}{K}$ which generates the bounded part [32-35]:

$$\underset{B^{rn}}{T} = \underset{B^{nr}}{T} = \underset{B^{nr,a}}{K^a}, \tag{38.a}$$

$$\underset{B^{sar}}{K} = -\left(q^2/4\right) w^{-4} \left[ w^{-1}(3-4W)\left(v_s \times k_a\right) k_r + 4\left(a_s \times k_a\right) k_r + g_{rs} k_a - g_{ra} k_s \right], \tag{38.b}$$

which means that $\underset{B^{ij}}{T}$ is the divergence of $\underset{B^{sar}}{K}$. This idea of the superpotential isn't Weert's original; it is actually quite old and was introduced by Freud [36] constructing the superpotential for the canonical energy-momentum pseudotensor of Einstein [37, 38]. Weert didn't study deeply the algebraic and differential properties of $\underset{B^{sar}}{K}$ which was remedied in [33, 39-41] obtaining a better comprehension of such superpotential structures:

$$\underset{B^{sar}}{K} = -\underset{B^{sar}}{K} \qquad \text{Antisymmetry}$$

$$\underset{B^{sr}}{K^r} = 0 \qquad \text{Null trace}$$

$$\underset{B^{sa,r}}{K^r} = 0 \qquad \text{Null divergence}$$

$$\underset{B^{sar}}{K} + \underset{B^{ars}}{K} + \underset{B^{rsa}}{K} = 0 \qquad \text{Cyclic} \tag{39}$$

Surprisingly, (39) is also satisfied in curved spaces (replacing partial derivatives with covariant ones) for the Lanczos spintensor $K_{sar}$ [42], which generates the Weyl conformal tensor in 4 dimensions [43-49]:

$$C_{jrim} = K_{jri;m} - K_{jrm;i} + K_{imj;r} - K_{imr;j} + g_{jm}K_{ir} - g_{ij}K_{mr} + g_{ri}K_{mj} - g_{rm}K_{ij}, \tag{40.a}$$

so that $K_{rj} = K_{jr} = K^a_{rj;a}$. This fact suggests at least two things:

1. The introduction in electrodynamics of the definition:

"A Minkowski spintensor is that which satisfies (39)",    (40.b)

so, the Weert superpotential is a Minkowskian spintensor.

2. To construct an "Electromagnetic Weyl tensor" through prescription (40.a) (in this case $\underset{B^{rj}}{K} = \underset{B^{rj}}{T}$ ):

$$\underset{B^{jrim}}{C} = \underset{B^{jri,m}}{K} - \underset{B^{jri,m}}{K} + \underset{B^{jrm,i}}{K} - \underset{B^{imj,r}}{K} - \underset{B^{imr,j}}{K} + g_{jm}\underset{B^{ir}}{T} - g_{ij}\underset{B^{mr}}{T} + g_{ri}\underset{B^{mj}}{T} - g_{rm}\underset{B^{ij}}{T} .$$    (40.c)

The Petrov classification [13] can be applied to $\underset{B^{jrim}}{C}$ , see [39, 41], resulting in Type II in the Penrose diagram, that is:

"The Liénard-Wiechert field is type II".    (40.d)

This strengthens the analogies found by Newman [50] between Robinson-Trautman metrics (Einstein's equations solution type II) [51] and the electromagnetic field of a point charge. The physical meaning of the Weert superpotential was elucidated in [40].

The idea (40.b) motivates the following question:

Can $\underset{B^{sar}}{K}$ be written as the sum of two or more Minkowskian spintensors?

The answer is affirmative because the terms in (38.b) can be grouped in the form [33]:

$$\underset{B^{sar}}{K} = \underset{B^{sar}}{\tilde{K}} + \underset{B^{sar}}{\bar{K}} ,$$    (41.a)

with:

$$\underset{B^{scr}}{\tilde{K}} = qw^{-2}\left[ qw^{-3}\left( v_s \times k_c \right) - F_{sc} \right]k_r ,$$    (41.b)

$$\underset{B^{sar}}{\bar{K}} = \left( q^2/4 \right)w^{-4}\left[ 3w^{-1}\left( v_a \times k_s \right)k_r + g_{ra}k_s - g_{rs}k_a \right] .$$    (41.c)

Both parts of $\underset{B^{sar}}{K}$ satisfy (39), so they are spintensors. By substituting (41.a) in (38.a) we obtain in a natural manner the splitting of López [52]:

$$\underset{B^{ra}}{T} = \underset{B^{ra}}{\tilde{T}} + \underset{B^{ra}}{\bar{T}} ,$$    (42.a)

where:

$$\underset{B^{rs}}{\tilde{T}} = \underset{B^{sr,a}}{\tilde{K}^a}, \qquad \underset{B^{rs}}{\overline{T}} = \underset{B^{sr,a}}{\overline{K}^a}, \tag{42.b}$$

so:

$$\underset{B^{r,s}}{\tilde{T}^s} = \underset{B^{r,s}}{\overline{T}^s} = 0. \tag{42.c}$$

Decomposition (42.a) is valuable in the study of electromagnetic angular momentum; here it came out as a consequence of the spintensor concept (40.b).

It can be proven that:

$$\underset{B^{sar}}{\overline{K}} = \left( \tfrac{q^2}{4} w^{-4} D^b_{sar} \right)_{,b} = (41.\text{c}), \tag{43.a}$$

where $D_{ijrm}$ is a tensor employed by Synge [7] in another context:

$$D_{sarb} = g_{rs}k_a k_b - g_{ab}k_r k_s - g_{ar}k_s k_b - g_{sb}k_a k_r; \tag{43.b}$$

the identity (43.a) was obtained by Rowe [53].

Weert didn't study (35.a): This analysis was considered in [54-61] to determine a non-local superpotential (it depends on integrals over the world line) for the radiative part:

$$\underset{R^{rs}}{T} = \underset{R^{sr,a}}{K^a}, \tag{44.a}$$

with:

$$\underset{R^{scr}}{K}\left( X^i \right) = qF_{sc}[ p_{(\sigma)} p_{(\theta)} \left( \int_0^\tau a^{(\sigma)} a^{(\theta)} v_r d\gamma + p_{(\beta)} \int_0^\tau a^{(\sigma)} a^{(\theta)} e_r^{(\beta)} d\gamma \right) -$$

$$\tag{44.b}$$

$$- \int_0^\tau a^2 v_r d\gamma - p_{(\sigma)} \int_0^\tau a^2 e_r^{(\sigma)} d\gamma], \qquad \sigma, \theta, \beta = 1,2,3$$

where $e^{(\sigma)}r$ is the Fermi triad and $\tau$ is the proper time in the retarded point associated to $X^i$. Trying out (44.a) brings into relevance the identities (24); the integrals in (44.b) indicate the non-local character of radiative superpotential; besides, if the 4-acceleration $a^r$ is annulated, then $\underset{R^{ijc}}{K} = 0$ which was to be expected due to (26). When obtaining (44), transport (9.a) is basic; never before had the great value of the Fermi triad been shown in electrodynamics.

## 6. CONCLUSIONS

It was possible to deduce non-local superpotential for the radiative part of the Maxwell tensor associated to the Liénard-Wiechert field. A Petrov classification was introduced of the

electromagnetic field produced by a charged particle in arbitrary motion, and it was proved that it has Type II. In our analysis, the Minkowskian spintensor concept was relevant because it enabled us to establish a connection between the Weert superpotential and the gravitational Lanczos potential for the Weyl tensor in general relativity. The participation of the Fermi triad was original and fundamental over the point charge trajectory.

## References

[1]   J. L. Synge, *Proc. Roy. Irish Acad*. A65 (1967) 27.

[2]   J. Stachel, *Phys. Rev*. 180 (1969) 1256.

[3]   C. Teitelboim, *Phys. Rev*. D1 (1970) 1572.

[4]   C. Teitelboim, *Phys. Rev*. D3 (1971) 297.

[5]   F. Miglietta, *J. Math. Phys*. 20 (1979) 868.

[6]   J. Plebañski, *The structure of the field of a point charge*, Internal Report, Cinvestav-IPN, Mexico city (1972).

[7]   J. L. Synge, *Ann. Mat. Pura Appl*. 84 (1970) 33.

[8]   P. A. M. Dirac, *Proc. Roy. Soc*. London A167 (1938) 148.

[9]   H. J. Bhabha, *Proc. Roy. Soc*. London A172 (1939) 384.

[10]  E. Fermi, *Atti. R. Accad. Lincei Rend*. 31 (1922) 21

[11]  F. A. E. Pirani, *Acta Phys. Polon. Sci*. Cl. 15 (1956) 389.

[12]  J. L. Synge, *Relativity: the general theory*, North-Holland, Amsterdam (1976).

[13]  M. Acevedo, J. López-Bonilla, *Gen. Rel. Grav*. 37(3) (2005) 627.

[14]  E. Piña, *Rev. Mex. Fís*. 16 (1967) 233.

[15]  G. Y. Rainich, *Trans. Amer. Math. Soc.* 27 (1925) 106.

[16]  J. Plebañski, *Bull. Acad. Polon. Sci*. Cl. 9 (1961) 587.

[17]  J. A. Wheeler, *Geometrodynamics*, Academic Press, New York (1962).

[18]  R. Penney, *J. Math. Phys*. 5 (1964) 1431.

[19]  C. Lanczos, *Ann. of Math*. 39 (1938) 842.

[20]  J. H. Caltenco, J. López-Bonilla, M. A. Martínez, A. Xeque M., *Aligarh Bull. Math.* 20 (2001) 61.

[21]  M. P. Drazin, *the Math. Gazz.* 36 (1952) 253.

[22]  E. S. Lowry, *Phys. Rev*. 117 (1960) 616.

[23]  V. Gaftoi, J. López-Bonilla, G. Ovando, *Aligarh Bull. Math.* 17 (1997-98) 59.

[24]  V. Gaftoi, J. López-Bonilla, G. Ovando, *Canad. J. Phys*. 79 (2001) 75.

[25]  J. López-Bonilla, J. Morales, G. Ovando, *Bull. Allahabad Math. Soc*. 17 (2002) 53.

[26]  A. Schild, *J. Math. Anal Appl*. 1 (1960) 127.

[27]  J. L. Synge, *Relativity: the special theory*, North-Holland, Amsterdam (1965).

[28]  Ch. G. van Weert, *Physica* 65 (1973) 452.

[29]  J. N. Goldberg, R. Kerr, *J. Math Phys*. 5 (1964) 172.

[30]  Ch. G. van Weert, *Phys. Rev*. D9 (1974) 339.

[31]  Ch. G. van Weert, *Physica* 76 (1974) 345.

[32]  V. Gaftoi, J. López-Bonilla, G. Ovando, *Int. J. Theor. Phys*. 38 (1999) 939.

[33]  J. López-Bonilla, J. Morales, G. Ovando, *Indian J Phys*. B74 (2000) 167.

[34]  J. López-Bonilla, G. Ovando, J. Rivera, *Indian J. Pure Appl. Math*. 28 (1997)1355.

[35]  V. Gaftoi, J. López-Bonilla, G. Ovando, *Nuovo Cim*. B114 (1999) 423.

[36]  Ph. von Freud, *Ann of Math.* 40 (1939) 417.

[37]  J. Caltenco, J. López-Bonilla, G. Ovando, *Indian J. Theor. Phys*. 51 (2003) 273.

[38]  J. H. Caltenco, J. López-Bonilla, R. Peña R., J. Rivera, *Proc. Pakistan Acad. Sci*. 42(4) (2005) 261.

[39]  N. Aquino, J. López-Bonilla, H. Núñez-Yépez, A.L. Salas-Brito, *J. Phys. A*: *Math. Gen*. 28 (1995) L375.

[40]  J. López-Bonilla, G. Ovando, J. Rivera, *Nuovo Cim*. B112 (1997) 1433.

[41]  J. H. Caltenco, J. López-Bonilla, R. Peña R., *Indian J. Theor. Phys*. 50 (2002) 1.

[42]  C. Lanczos, *Rev. Mod. Phys*. 34 (1962) 379.

[43]  V. Gaftoi, J. López-Bonilla, G. Ovando, *Nuovo Cim*. B113 (1998) 1489.

[44]  J. López-Bonilla, J. Morales, G. Ovando, *Gen. Rel. Grav*. 31 (1999) 413.

[45]  J. López-Bonilla, G. Ovando, J. Peña, *Found. Phys. Lett.* 12 (1999) 401.

[46]  J. López-Bonilla, J. Morales, G. Ovando, J. Rivera, *Indian J. Theor. Phys*. 48 (2000) 289.

[47]  J. H. Caltenco, J. López-Bonilla, J. Morales, G. Ovando, *Chinese J. Phys*. 39 (2001) 397.

[48]  M. Acevedo, J. López-Bonilla, S. Vidal-Beltrán, *Grav. & Cosm*. 10 (2004) 328.

[49]  J. H. Caltenco, R. Linares y M., J. López-Bonilla, *Proc. Pakistan Acad. Sci*. 42(2) (2005) 153.

[50]  E. Newman, *J. Math. Phys*. 15 (1974) 44.

[51]  J. López-Bonilla, J. Rivera, *Indian J. Math*. 40 (1998) 159.

[52]  C. A. López, *Phys. Rev*. D17 (1978) 2004.

[53]  E. G. P. Rowe, *Phys. Rev*. D18 (1978) 3639.

[54] N. Aquino, O. Chavoya, J. López-Bonilla, D. Navarrete, *Nuovo Cim*. B108 (1993) 1081.

[55] J. López-Bonilla, J. Morales, M. Rosales, *Pramana J. Phys*. 42 (1994) 89.

[56] J. López-Bonilla, H. Núñez-Yépez, A.L. Salas-Brito, *J. Phys. A: Math. Gen*. 30 (1997) 3663.

[57] J. López-Bonilla, G. Ovando, *Gen. Rel. Grav*. 31 (1999) 1931.

[58] J. López-Bonilla, G. Ovando, *Indian J. Theor. Phys*. 49 (2001) 139.

[59] J. H. Caltenco, J. López-Bonilla, J. Morales, G. Ovando, *Chinese J. Phys*. 40 (2002) 214.

[60] M. Acevedo, J. López-Bonilla, J. Sosa, *Apeiron* 9(1) (2002) 43.

[61] V. Barrera, J. López-Bonilla, J. Sosa, *Comm. in Phys*. (Viet Nam) 19(4) (2009) 229.