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Approximate Bound State Solution of Relativistic Klein-Gordon Particles with Physical Potentials

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ABSTRACT

This dissertation presents an Approximate Bound State Solution of Relativistic Klein-Gordon Particles with Physical Potentials. We have solved the approximate bound state solution of the Klein-Gordon equation for unequal scalar and vector Hulthen potentials and Modified Hellmann plus Hylleraas potential for arbitrary l – state. We have used the parametric generalization of the Nikiforov-Uvarov method to obtain the bound state energy eigenvalues and the corresponding wave function expressed in term of the Jacobi Polynomials. The energy eigenvalue for these two potentials were computed using different screening parameter and the numerical variation of these potentials with the radial distance between the interacting particles are shown in Table 3 and 4. The approximation scheme used in this work were compared with the centrifugal term and observed that the approximation is preferred for small screening parameters, showing that these potential are short range.

***Keywords*:** The Klein-Gordon, Approximate Bound State, Eigen Value, Nikiforov-Uvarov method

1. BACKGROUND OF STUDY

One major area of interest in high energy and particle physics is obtaining the problems of exact solution of the Klein-Gordon, Duffin-Kemmer-Petiau and Dirac equations for mixed vector and scalar potentials. When a particle is in a strong potential field, the relativistic effect must be considered, which gives the correction for non-relativistic quantum mechanics

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(Oyewumi et al2010). The Klein-Gordon, Dirac, and Duffin-Kemmer-petiau wave equations are frequently used to describe the particle dynamics in relativistic quantum mechanics.

In relativistic quantum mechanics, one can apply the Klein-Gordon equation to the treatment of a zero-spin particle and the Dirac equation for spin half particle. In recent years, many studies have been carried out to explore the relativistic energy eigen values and corresponding wave functions of the Klein-Gordon and Dirac equations (Jia et al, 2006).

These relativistic equations contain two objects: the four – vector linear momentum operation and the scalar rest mass. These allow one to introduce two types of potential coupling, which are the four vector potential V(r), and the space – time scalar potential S(r).

The Klein–Gordon equation with the vector and scalar potentials can be written as follows (Ikot et al 2012):

$$\left[-\left(i\hbar\frac{\partial}{\partial t}-V(r)\right)^2-\nabla^2+(S(r)+M)^2\right]\Psi(r,\theta,\varphi)=0 \qquad 1$$

where: M is the rest mass $i\hbar\frac{\partial}{\partial t}=E$ is the energy eigenvalues, V(r) and S(r) are the vector and scalar potentials respectively.

For example, some authors have assumed that the scalar potential is equal to the vector potential and obtained the exact solutions of the Klein - Gordon equation and Dirac equation with some typical potential using different methods (Ikot et al, 2011). For examples, some authors have assumed that the scalar potential is equal to the vector potential and obtained the exact solutions of the Klein – Gordon equation and Dirac equation with some typical potential using different methods. For example, these investigations have employed anharmonic oscillator potential, Kratzer potential (Qiang, 2004), ring shape pseudoharmonic potential (Ikhadair et al 2008), Woods – Saxon potential (Berkdemir et al 2006), Scarf potential (Zhang et al 2005), Hartmann potential (De Souza et al 2006), Poschl-Teller potential and Rosen Morse Potential (Alhaidari, 2001). Different methods such as the Supersymmetric Quantum Mechanics, Asymptotic iteration method (AIM) (Ciftcim et al 2003) and Nikiforov-Uvarov (NU) (Cheng et al 2007) and others have been used to solve the second order differential equations arising from these considerations (Louis et all, V. 70, 74, 77, 2017).

However, the analytical solutions of the Klein – Gordon equations are possible only in the s – wave case with the angular momentum l = 0 for some exponential type potential models (Diao et al 2004). Conversely, when l = 0, one can only solve approximately the Klein – Gordon equation and the Dirac equation for some potential using a suitable approximation scheme (Xu, et al 2010).

2. METHODOLOGY

2. 1. Factorization Method

The time-independent Klein-Gordon equation with the scalar potential S(r) and vector potential V(r) in the natural unit ($\hbar = c = 1$) is given as

$$[\nabla^2+(V(r)-E)^2-(S(r)+M)^2]\psi(r,\theta,\varphi)=0 \qquad 2$$

where: M is the rest mass, E is the relativistic energy, ∇^2 is the Laplace operator. In the spherical coordinates, the Klein-Gordon equation for a particle in the present of a modified potential V(r) becomes

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}-2\big(EV(r)+MS(r)\big)+V^2(r)-S^2(r)+E^2-M^2\right]\times\psi(r,\theta,\varphi)=0 \qquad 3$$

If one assign the total wave function as

$$\psi(r,\theta,\varphi)=\frac{R(r)}{r}Y_{lm}(\theta,\varphi) \qquad 4$$

Eq. 3 is separated into variables and the following equations are obtained:

$$\frac{d^2R(r)}{dr^2}\left[E^2-M^2-2\big(EV(r)+MS(r)\big)+V^2(r)-S^2(r)-\frac{\lambda}{r^2}\right]R(r)=0 \qquad 5$$

$$\frac{d^2\Theta(\theta)}{d\theta^2}+\sin\theta\frac{d\Theta(\theta)}{d\theta}+\left(\lambda-\frac{m^2}{\sin^2\theta}\right)\Theta(\theta)=0 \qquad 6$$

$$\frac{d^2\Phi(\varphi)}{d\varphi^2}+m^2\Phi(\varphi)=0 \qquad 7$$

where: $Y_{lm}(\theta,\varphi)=\Theta(\theta)\Phi(\varphi)$ 8

and m and $\lambda=l(l+1)$ are the separation constants. Eq. (4) and Eq. (5) are spherical harmonic function $Y_{lm}(\theta,\varphi)$ whose solutions are well known. The above equations can be used to solve radial Klein-Gordon equation for different potentials

2. 2. Solutions of K.G.E with Unequal Scalar and Vectors Hulthen Potentials

One of the significant short-range potentials is the Hulthen potential. It has been widely used in the solid state physics, atomic physics, nuclear and particle physics and chemical physics. Hulthen potential in the non-relativistic and the relativistic quantum mechanics has been studied recently. Thus, it is worth to investigate the solution of the Klein-Gordon equation for unequal vector and scalar Hulthen potentials respectively

We have decided to use the vector and scalar potential respectively as:

$$V(r)=\frac{-V_0e^{-2\propto r}}{1-e^{-2\propto r}} \qquad 9$$

$$S(r)=\frac{-S_0e^{-2\propto r}}{1-e^{-2\propto r}} \qquad 10$$

Therefore, we take the approximation scheme given by (Hamzavi et al., 2012) to deal with the centrifugal term

$$\frac{1}{r^2} \approx 4\alpha^2 \frac{e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2} \qquad 11$$

Putting Eq. (186), (187) and (188) into (189), we obtain

$$\frac{d^2R}{dr^2} + \left[\begin{matrix} E^2 - M^2 - 2\left\{\frac{-EV_0 e^{-2\propto r}}{1-e^{-2\propto r}} - \frac{MS_0 e^{-2\propto r}}{1-e^{-2\propto r}}\right\} + \\ \frac{V_0^2 e^{-4\propto r}}{(1-e^{-2\propto r})^2} - \frac{S_0^2 e^{-4\propto r}}{(1-e^{-2\propto r})^2} - 4\lambda \propto^2 \left[\frac{e^{-2\propto r}}{(1-e^{-2\propto r})^2}\right]\end{matrix}\right] R(r) = 0 \qquad 12$$

To convert equation (13) to a hypergeometric equation, we make use of the

transformation; $S = e^{-2\propto r}$ 13

$$\frac{d^2R}{dr^2} = \frac{d}{dr}\left[\frac{dR}{ds}\frac{ds}{dr}\right] \qquad 14$$

$$\frac{dS}{dr} = -2 \propto e^{-2\propto r} \qquad 15$$

$$\frac{d}{dr} = -2 \propto e^{-2\propto r}\frac{d}{ds} \qquad 16$$

Putting Eq. (193) and (194) into (192) we obtained;

$$\frac{d^2R}{dr^2} = 4 \propto^2 e^{-2\propto r}\frac{dR}{ds} + 4 \propto^2 e^{-4\propto r}\frac{d^2R}{ds^2} \qquad 17$$

Using Eq. (14) we have

$$\frac{d^2R}{dr^2} = 4 \propto^2 s\frac{dR}{ds} + 4 \propto^2 s^2\frac{d^2R}{ds^2} \qquad 18$$

Using Eq. (14) to deal with the second term we obtained;

$$\left[\begin{matrix} E^2 - M^2 + 2\left\{EV_0\frac{s}{(1-s)} - MS_0\frac{s}{(1-s)}\right\} + V_0^2\frac{s^2}{(1-e^{-2\alpha r})^2} \\ -S_0^2\frac{s^2}{(1-s)^2} - 4\lambda \propto^2\left[\frac{s}{(1-s)^2}\right]\end{matrix}\right] R(s) = 0 \quad ` \qquad 19$$

$$\frac{4\alpha^2}{(1-s)^2}\left[\begin{matrix} E^2\frac{(1-s)^2}{4\alpha^2} - M^2\frac{(1-s)^2}{4\alpha^2} - \frac{1}{2\alpha^2}\left[MS_0 s(1-s) - EV_0 s(1-s)\right] \\ +V_0^2\frac{s^2}{4\alpha^2} - S_0^2\frac{s^2}{4\alpha^2} - \lambda s \end{matrix}\right] \qquad 20$$

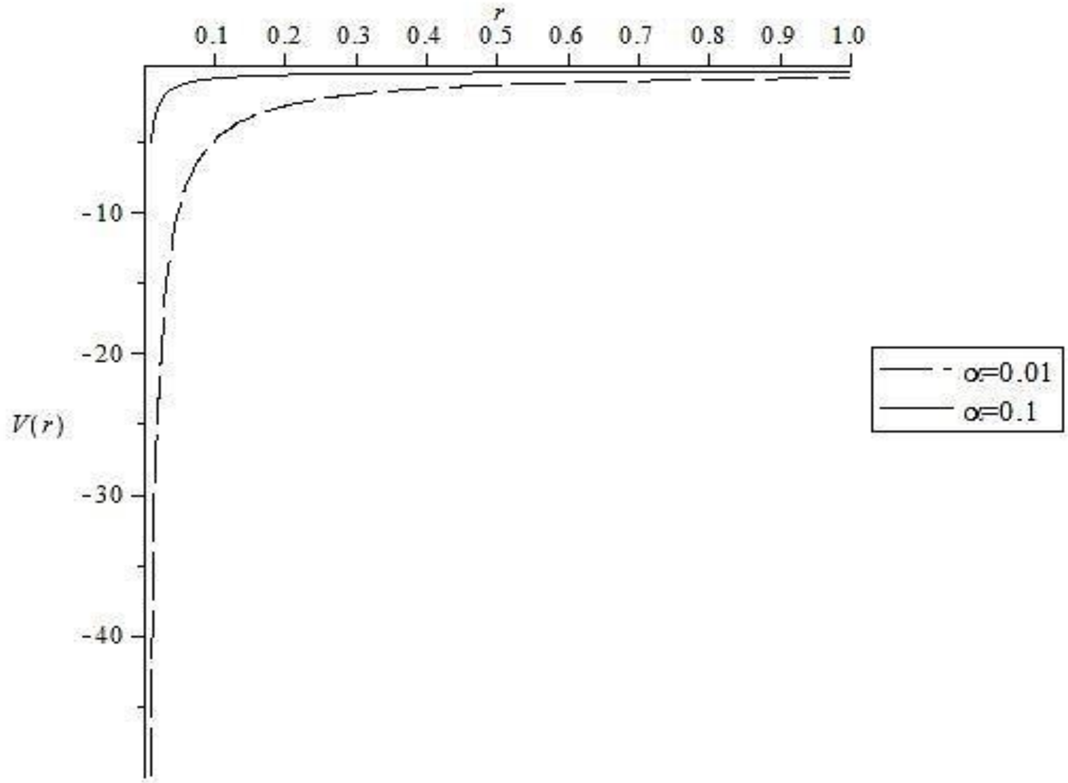

Figure 1. Variation of Unequal Scalar and Vectors Hulthen Potentials with r for V_0 = 0.01MeV for values of α = 0.01 and 0.1

Evaluating Eq. (20) we have;

$$\frac{4\alpha^2}{(1-s)^2}\left[-\left(\frac{-E^2}{4\alpha^2}+\frac{M^2}{4\alpha^2}-\frac{MS_0}{2\alpha^2}+\frac{EV_0}{2\alpha^2}-\frac{V_0^2}{4\alpha^2}+\frac{S_0^2}{4\alpha^2}\right)s^2+\left(\frac{-E^2}{2\alpha^2}+\frac{M^2}{2\alpha^2}-\frac{MS_0}{2\alpha^2}+\frac{EV_0}{2\alpha^2}-\lambda\right)s-\left(\frac{-E^2}{4\alpha^2}+\frac{M^2}{4\alpha^2}\right)\right] \quad 21$$

Combining Eq. (20) and (21) we have;

$$4\alpha^2 s^2\frac{d^2R}{ds^2}+4\alpha^2 s\frac{dR}{ds}+\frac{4\alpha^2}{(1-s)^2}\left[-\left(\frac{-E^2}{4\alpha^2}+\frac{M^2}{4\alpha^2}-\frac{MS_0}{2\alpha^2}+\frac{EV_0}{2\alpha^2}-\frac{V_0^2}{4\alpha^2}+\frac{S_0^2}{4\alpha^2}\right)s^2+\left(\frac{-E^2}{2\alpha^2}+\frac{M^2}{2\alpha^2}-\frac{MS_0}{2\alpha^2}+\frac{EV_0}{2\alpha^2}-\lambda\right)s-\left(\frac{-E^2}{4\alpha^2}+\frac{M^2}{4\alpha^2}\right)\right] \quad 22$$

Dividing (22) by $4\alpha^2 s^2$ obtained;

$$\frac{d^2R}{ds^2}+\frac{(1-s)}{s(1-s)}\frac{dR}{ds}+\frac{1}{s^2(1-s)^2}\left[\begin{array}{l}-\left(\frac{-E^2}{4\alpha^2}+\frac{M^2}{4\alpha^2}-\frac{MS_0}{2\alpha^2}+\frac{EV_0}{2\alpha^2}-\frac{V_0^2}{4\alpha^2}+\frac{S_0^2}{4\alpha^2}\right)s^2+\\ \left(\frac{-E^2}{2\alpha^2}+\frac{M^2}{2\alpha^2}-\frac{MS_0}{2\alpha^2}+\frac{EV_0}{2\alpha^2}-\lambda\right)s_{-}\left(\frac{-E^2}{4\alpha^2}+\frac{M^2}{4\alpha^2}\right)\end{array}\right] \quad 23$$

Comparing Eq. (23) with Eq. (4) we obtained;

$$c_1 = c_2 = c_3 = 1$$

Evaluating Eq. (7) we obtain the following parameters

$$\begin{array}{l} c_4=0, c_5=\frac{-1}{2}, c_6=\frac{1}{4}-\frac{E^2}{4\alpha^2}+\frac{M^2}{4\alpha^2}-\frac{MS_0}{2\alpha^2}+\frac{EV_0}{2\alpha^2}-\frac{V_0^2}{4\alpha^2}+\frac{S_0^2}{4\alpha^2}, \\ c_7=\frac{E^2}{2\alpha^2}-\frac{M^2}{2\alpha^2}+\frac{MS_0}{2\alpha^2}-\frac{EV_0}{2\alpha^2}+\lambda \\ c_8=-\frac{E^2}{4\alpha^2}+\frac{M^2}{4\alpha^2}=Y \\ c_9=\frac{1}{4}-\frac{V_0^2}{4\alpha^2}+\frac{S_0^2}{4\alpha^2}+\lambda=X \\ c_{10}=1+2\sqrt{Y}, c_{11}=2+2\left(\sqrt{X}+\sqrt{Y}\right), c_{12}=\sqrt{Y}, c_{13}=-\frac{1}{2}-\left(\sqrt{X}+\sqrt{Y}\right) \end{array} \quad 24$$

Substituting Eq. (24) into Eq. (14) we have;

$$\begin{array}{l} n^2+n+\frac{1}{2}+(2n+1)\left(\sqrt{X}+\sqrt{Y}\right)+\frac{E^2}{2\alpha^2}-\frac{M^2}{2\alpha^2}+\frac{MS_0}{2\alpha^2}-\frac{EV_0}{2\alpha^2}+ \\ \lambda-\frac{E^2}{2\alpha^2}+\frac{M^2}{2\alpha^2}+2\sqrt{XY}=0 \end{array} \quad 25$$

Evaluating Eq. (25) explicitly we have;

$$n^2+n+\frac{1}{2}+(2n+1)\left(\sqrt{X}+\sqrt{Y}\right)+\frac{MS_0}{2\alpha^2}-\frac{EV_0}{2\alpha^2}+\lambda+2\sqrt{XY}=0 \quad 26$$

Letting $n^2+n+\frac{1}{2}+(2n+1)\sqrt{X}+\frac{MS_0}{2\alpha^2}+\lambda=\gamma$

$$(2n+1)\sqrt{Y}+2\sqrt{XY}=\frac{EV_0}{2\alpha^2}-\gamma \qquad 27$$

Squaring both sides of Eq. (27) we have;

$$Y\left((2n+1)+2\sqrt{X}\right)=\frac{E^2V_0^2}{4\alpha^4}+\gamma^2-\frac{EV_0}{\alpha^2}\gamma \qquad 28$$

$$\left(-\frac{E^2}{4\alpha^2}+\frac{M^2}{4\alpha^2}\right)\left((2n+1)+2\sqrt{X}\right)^2=\frac{E^2V_0^2}{4\alpha^4}+\gamma^2-\frac{EV_0}{\alpha^2}\gamma \qquad 29$$

Carrying out simple algebra we obtained;

$$E^2\left(\frac{\left((2n+1)+2\sqrt{X}\right)^2}{4\alpha^2}+\frac{V_0^2}{4\alpha^4}\right)-E\left(\frac{V_0}{\alpha^2}\gamma\right)-\frac{1}{4}\left(\frac{M^2}{\alpha^2}\left((2n+1)+2\sqrt{X}\right)^2+4\gamma^2\right)=0 \qquad 30$$

Eq. (30) is a quadratic equation and can be solved using the quadratic formula. The energy eigenvalue equation for the Klein-Gordon equation for the case of unequal scalar and vector potential with constant mass is given as;

$$E=\frac{\left(\frac{V_0}{\alpha^2}\gamma\right)\pm\sqrt{\left(\frac{V_0}{\alpha^2}\gamma\right)^2+\varepsilon}}{\left(\frac{\left((2n+1)+2\sqrt{X}\right)^2}{2\alpha^2}+\frac{V_0^2}{2\alpha^4}\right)} \qquad 31$$

where: $\varepsilon=\left(\frac{\left((2n+1)+2\sqrt{X}\right)^2}{\alpha^2}+\frac{V_0^2}{\alpha^4}\right)\left(\frac{M^2}{\alpha^2}\left((2n+1)+2\sqrt{X}\right)^2+4\gamma^2\right)$

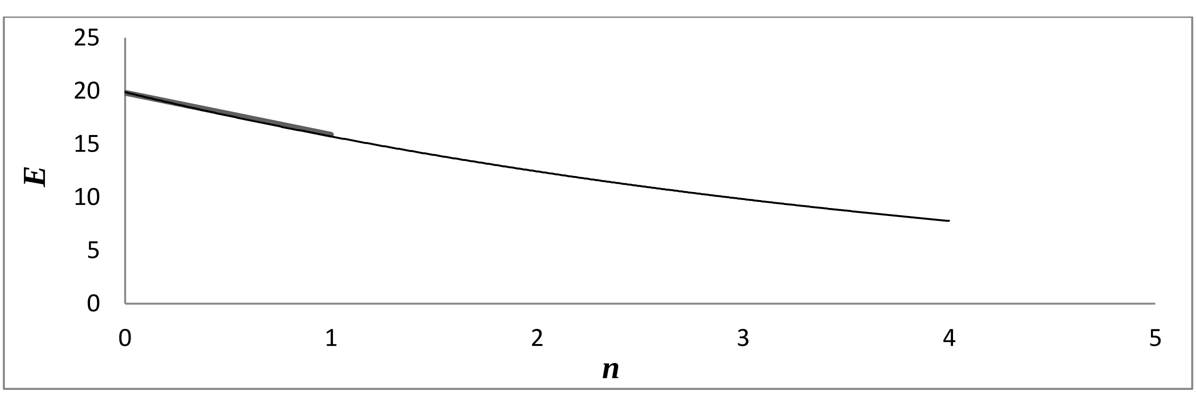

Fig. 2. Variation of the energy eigenvalues for Unequal Scalar and Vectors Hulthen Potential against *n*

Evaluating Eq. (31)

$$\psi(s) = N_{nl} s^{\sqrt{Y}} \left(1 - s^{\frac{1}{2}+\sqrt{X}}\right) P_n^{\left[2\sqrt{Y};2\sqrt{X}\right]}(1-2s) \quad 32$$

Using the transformation in Eq. (210) we have

$$\psi(r) = N_{nl} e^{-2\alpha r\sqrt{Y}} \left(1 - e^{-2\alpha r\left(\frac{1}{2}+\sqrt{X}\right)}\right) P_n^{\left[2\sqrt{Y};2\sqrt{X}\right]}\left(1-2e^{-2\alpha r}\right) \quad 33$$

The total wave function is given by

$$\psi(r,\theta,\varphi) = N_{nl} e^{-2\alpha r\sqrt{Y}} \left(1 - e^{-2\alpha r\left(\frac{1}{2}+\sqrt{X}\right)}\right) P_n^{\left[2\sqrt{Y};2\sqrt{X}\right]}\left(1-2e^{-2\alpha r}\right) Y_{lm}(\theta,\varphi) \quad 34$$

3. DISCUSSION OF RESULTS

The variation of the Unequal Scalar and Vectors Hulthen Potentials with the radial distance of separation between the interacting particles for screening parameters $\alpha = 0.1$ and 0.01 is presented in Table 1 and the graphical variation is shown in Figure 12, 13, and 14.

The energy spectrum of the Unequal Scalar and Vectors Hulthen Potentials is reported numerically for various states with two different screening parameters $\alpha = 0.01$ and 0.1 in Table 2. Where the relativistic energy were obtained. From our literature review, equation 11 shows the energy spectrum for the deformed Hulthen potential with position dependent mass, the energy eigenvalues and the unnormalized wave function expressed in terms of the Jacobi polynomials when $\alpha = 1$, while the Unequal Scalar and Vectors Hulthen Potentials that we have worked on gave a screening parameters as small as $\alpha = 0.1$

Considering a generalized Tietz-Wei (GTW) potential defined by the Mobius square potential as

$$\mathrm{V(r)} = \mathrm{D_e}\left(\frac{\mathrm{A+Be^{-2\alpha r}}}{\mathrm{C+D'e^{-2r}}}\right)^2 \quad 35$$

Ikot et al (2013), used an approximation for a short range potential of the form

$$\frac{1}{r^2} \approx 4\alpha^2 \frac{C^2}{(C+D'e^{-2r})^2} \quad 36$$

In order to test for the accuracy and validity of this approximation, the plot in Figure 3 shows that the centrifugal term is a good approximation for a short range potential.

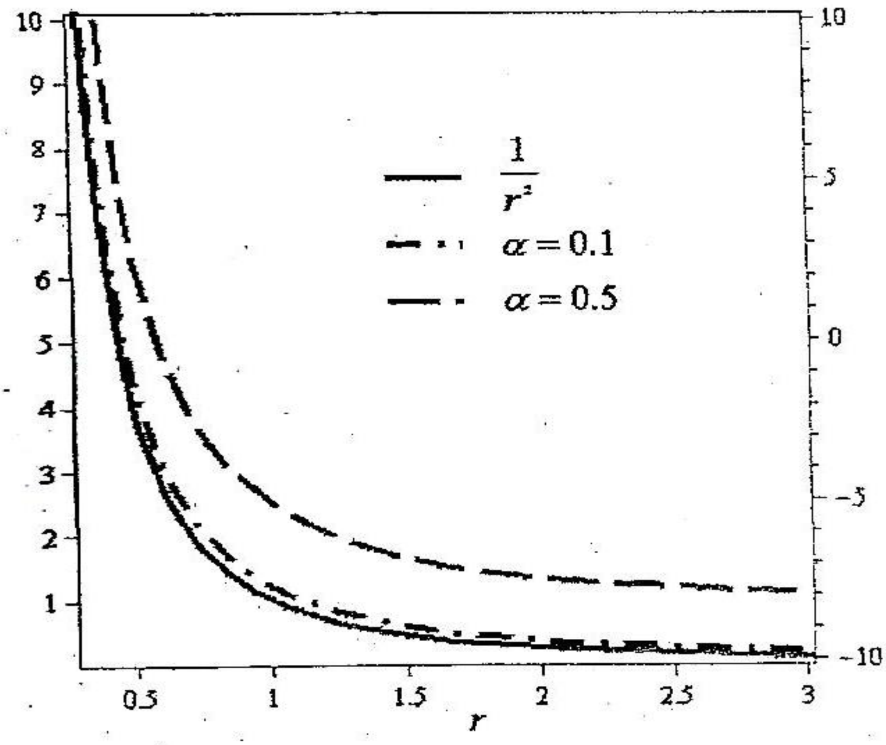

Figure 3. $\frac{1}{r^2}$ and its approximation $(C = 1, D' = -1\ )$

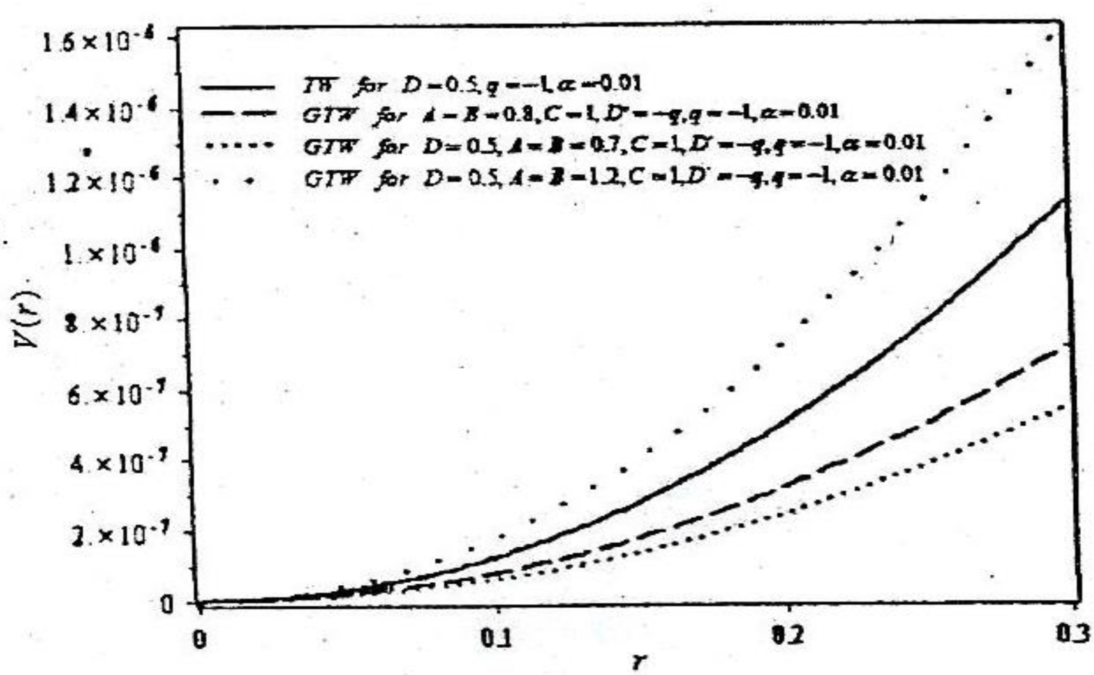

Figure 4. shapes of TW and GTW potentials

The variation of the Equal Scalar and Vector Modified Hellman plus Hylleraas Potential with the radial distance of separation between the interacting particles for screening parameters $\alpha = 0.1$and 0.01 is presented in Table 2 and the graphical variation is shown in Figure 4.

From the literature review, (Antia et al 2012) worked on unequal scalar and vector modified Hylleraas potential, where the modified Hylleraas Scalar and vector potentials are

$$S(r) = S_o\left(\frac{a+e^{\alpha r}}{e^{\alpha r}-1}\right), \qquad V(r) = V_o\left(\frac{a+e^{\alpha r}}{e^{\alpha r}-1}\right) \qquad 37$$

respectively and choosing a proper approximation to centrifugal term as

$$\frac{1}{r^2} = \propto^2\left[\frac{1}{12} + \frac{e^{\alpha r}}{(e^{\alpha r}-1)^2}\right] \qquad 38$$

The plot of the approximation of equation 38 and compared with the centrifugal term $\frac{1}{r^2}$ for three values of $\propto = 0.3, 0.4, 0.5$ respectively is shown in Figure 5 below.

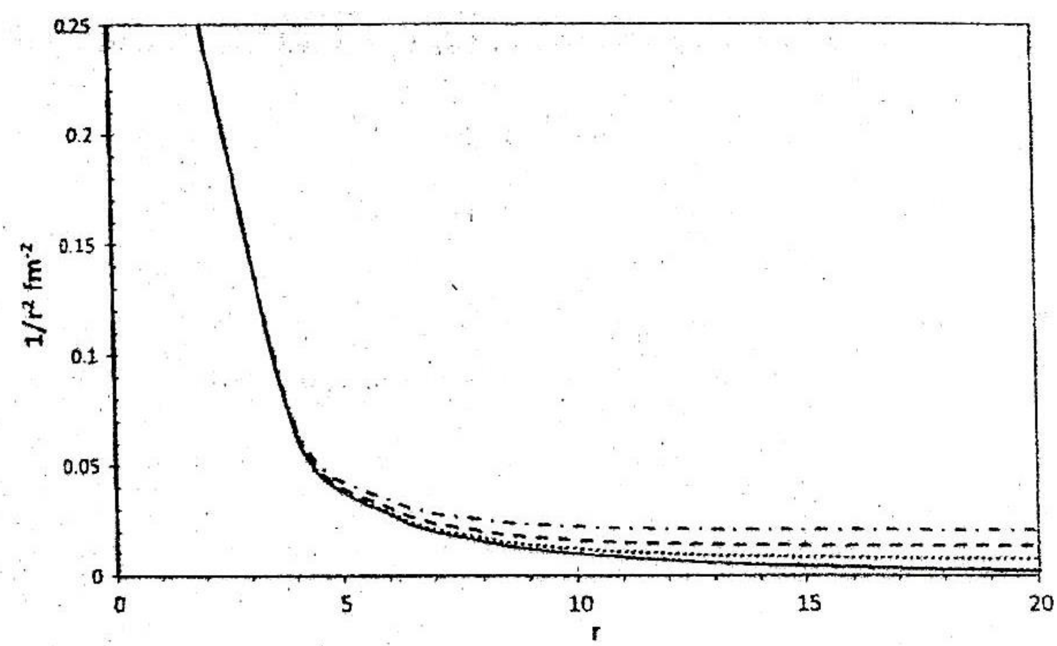

Figure 5. A plot of the variation of the centrifugal term $\frac{1}{r^2}$ and the approximation as a function of r for various values of $\propto = \mathbf{0.3, 0.4, 0.5}\ \boldsymbol{fm^{-1}}$

The energy spectrum of the Equal Scalar and Vector Modified Hellman plus Hylleraas Potential is reported numerically for various states with two different screening parameters $\alpha = 0.01$and 0.01 in Table 2. Where the relativistic energy were obtained. In order to solve the bound state solutions of d-dimensional Schrodinger equation with Eckart potential plus modified deformed Hylleraas potential (Ikot et al 2013) invoke an approximation for the centrifugal term

$$\frac{1}{r^2} = \frac{\alpha^2 e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \qquad 39$$

This approximation is good for small values of the parameter $\propto$ but it not applicable for large value to accommodate large values for $\propto$, (Ikot et al 2013) used a newly improved approximation scheme that is

$$\frac{1}{r^2} = \frac{\omega e^{-2\alpha r}}{1-e^{-2\alpha r}} + \frac{\lambda e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \qquad 40$$

where: ω and λ are adjustable dimensionless parameters.

The potential Eckar plus Hylleraas was defined as

$$V(r) = \frac{V_o}{b}\left(\frac{a-e^{-2\alpha r}}{1-e^{-2\alpha r}}\right) - V_1 \frac{e^{-2\alpha r}}{1-e^{-2\alpha r}} + V_2 \frac{e^{-2\alpha r}}{(1-e^{-2\alpha r})^2} \qquad 41$$

where: $V_o, V_1,$ and V_2 are the depths of the potential well, a and b Hylleraaas parameters and $\propto$ is the inverse of the range of the potential. (Ikot et al 2013) were able to obtain plots of the deformed Hyllerass potential, Eckart potential and the combined potentials $V(r)$ of equation (41) respectively.

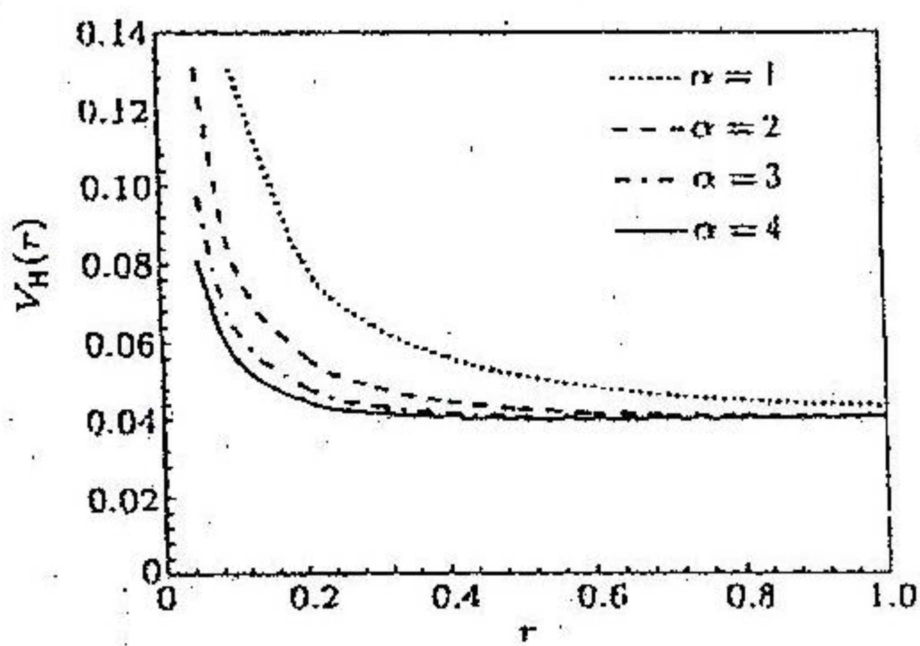

Figure 6. Modified deformed Hylleraas potentials versus r with $a = 2.0, b = 50.0,$ and $V_o = 1.0MeV, a = 2.0,$ and $b = 50.0$ for parameter $\propto = 1, 2, 3,$ and $4.$

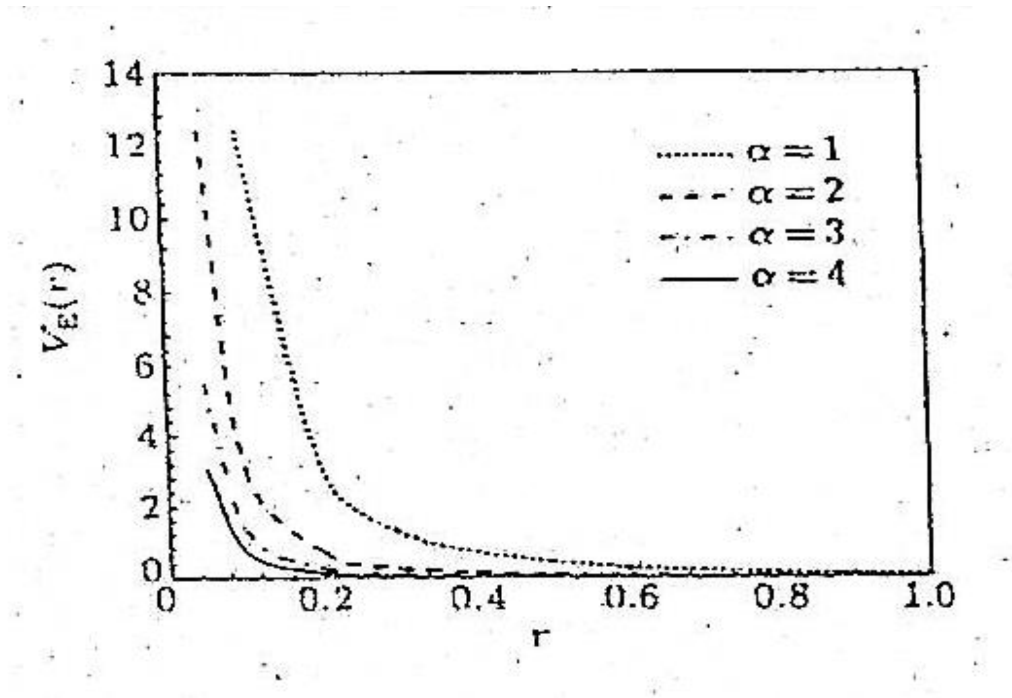

Figure 7. Eckart potentials versus r with
$V_1 = 0.01$ MeV and $V_2 = 0.5$ MeV for parameter $\propto = 1, 2, 3,$ and 4

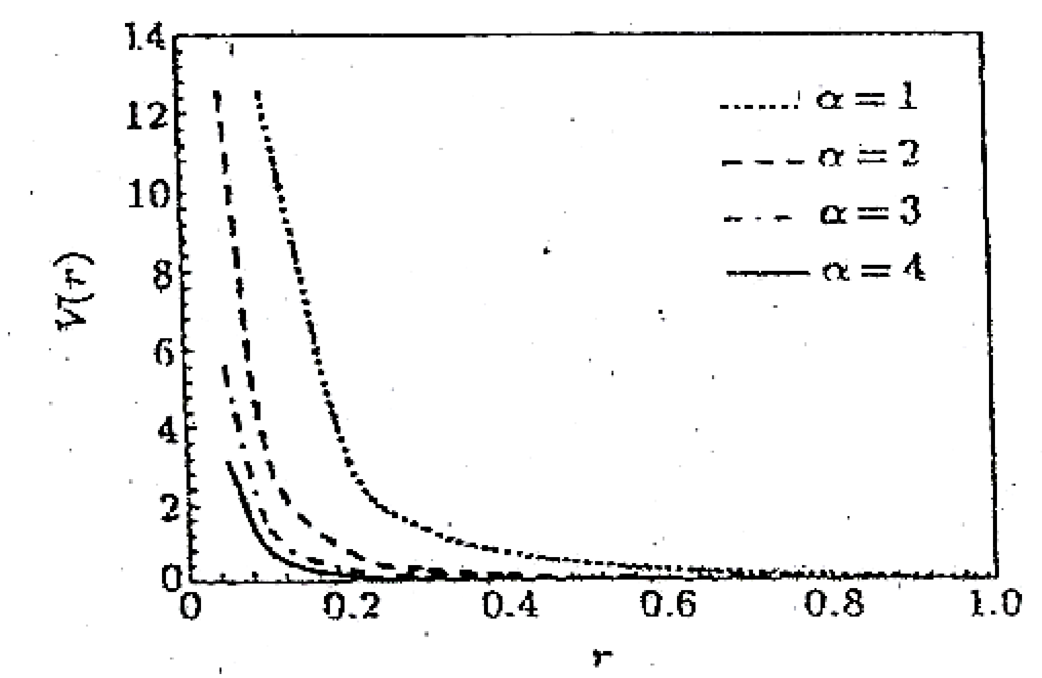

Figure 8. Deformed Hylleraas plus Ekart potentials versus r with
$V_1 = 0.01$ MeV and $V_2 = 0.5$ MeV $a = 2.0,$ and $b = 50.0$ for parameter $\propto = 1, 2, 3,$ and 4

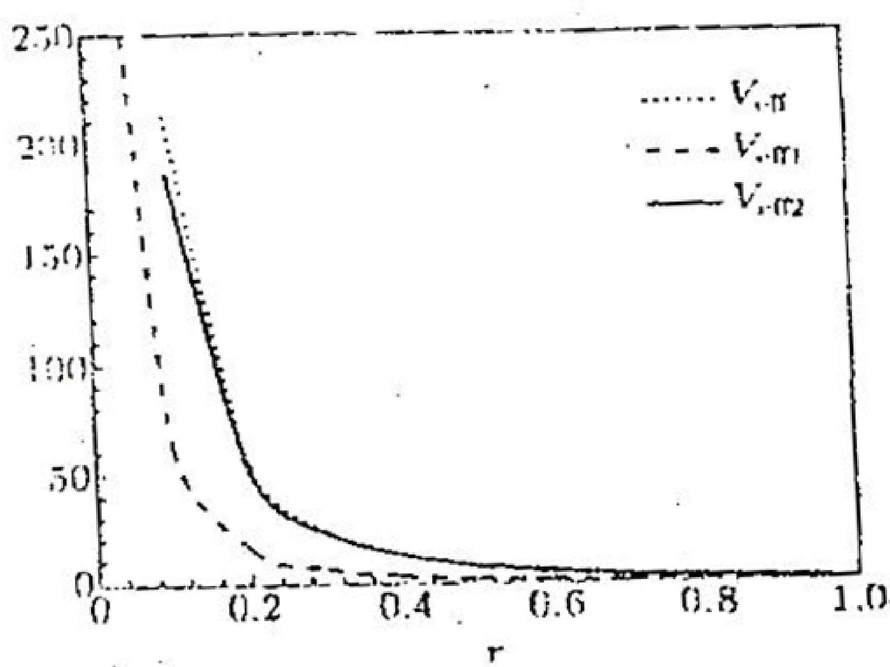

Figure 9. Effective potentials ($V_{eff}$for $1/r^2$, V_{eff1} for the approximation in equation (33) and V_{eff2} for the approximate in equation (241) versus r with $V_o = 1.0MeV$, $V_1 = 0.01MeV$, $V_2 = 0.5$ MeV, $a = 2.0, b = 50.0, l = 1, \propto = 1, \lambda = 3.2$ and $\omega = 1.6$

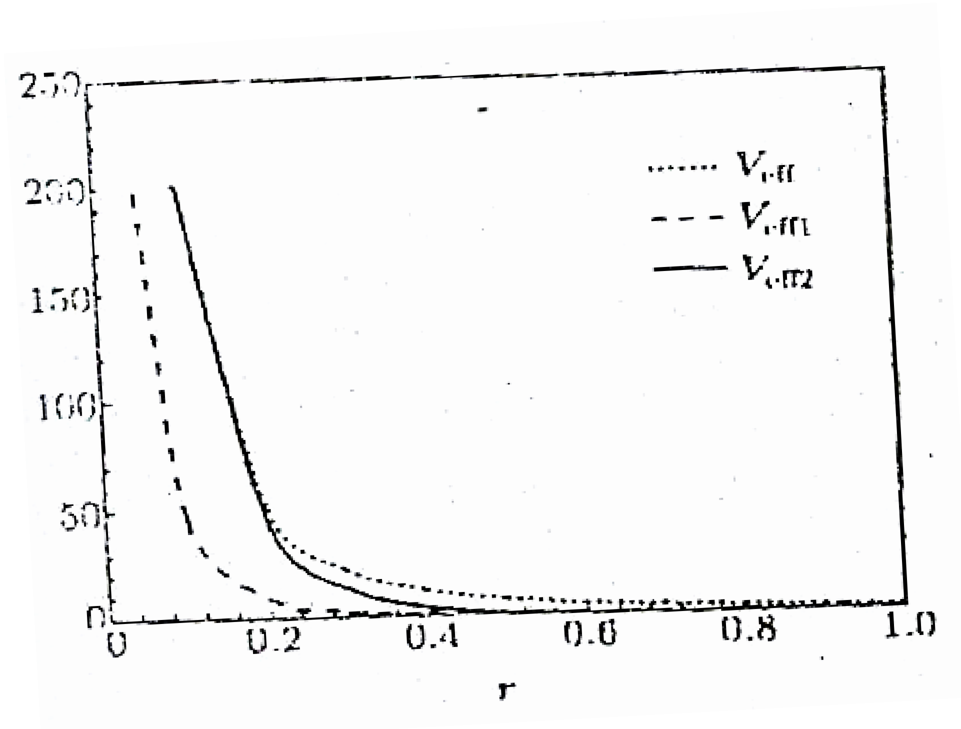

Figure 10. Effective potentials ($V_{eff}$for$1/r^2$, V_{eff1} for the approximation in equation (33) and V_{eff2} for the approximate in equation (241) versus r with $V_o = 1.0MeV$, $V_1 = 0.01MeV$, $V_2 = 0.5$ MeV, $a = 2.0, b = 50.0, l = 1, \propto = 5, \lambda = 3.3$ and $\omega = 1.7$

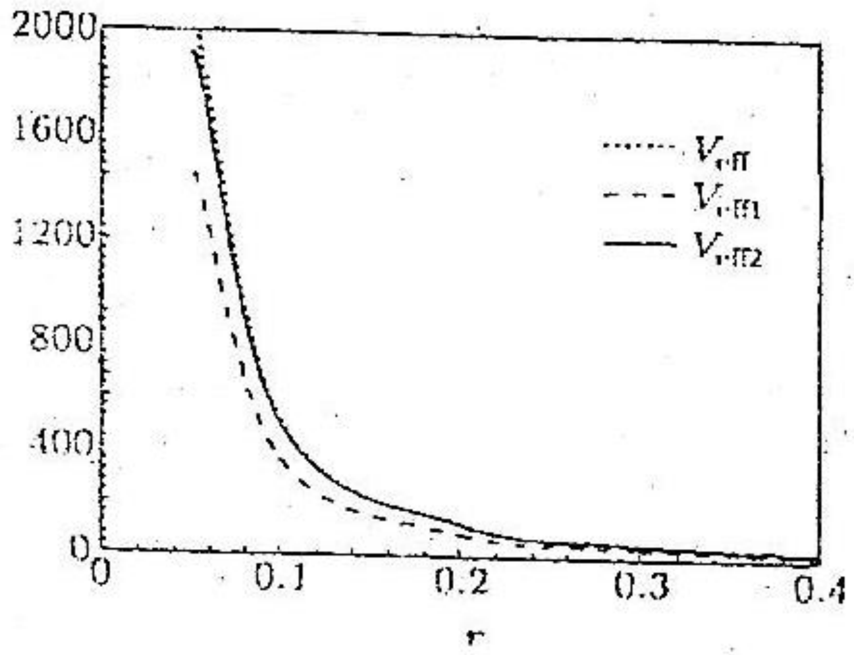

Figure 11. Effective potentials ($V_{eff}$for$1/r^2$, V_{eff1} for the approximation in equation (33) and V_{eff2} for the approximate in equation (241) versus r with $V_o = 1.0\text{MeV}$, $V_1 = 0.01\text{MeV}$, $V_2 = 0.5$ MeV, $a = 2.0, b = 50.0, l = 1, \propto = 0.2, \lambda = 3.1$ and $\omega = 12$

Compared to the energy spectrum for unequal scalar and vector modified Hylleraas potential obtained by (Antia et al2012), we have seen that the Equal Scalar and Vector Modified Hellman plus Hylleraas Potential is good for the screening parameter for $\alpha = 1, 2$ and 3.

Where $V_0 = 0.01\text{MeV}, V_1 = 0.02\text{MeV}, V_2 = 0.03\text{MeV}, V_3 = 0.04\text{MeV}, a = d = 2,$ and $b = 1$. The plot, figure 3, 4 and 5 shows the graph of the V(r) against r for the Equal Scalar and Vector Modified Hellman plus Hylleraas Potential. While for unequal scalar and vector Hulthen potential, the screening parameters where $\alpha = 0.01,$ and 0.1 respectiviely. Figure 1 shows the graph of the potential V(r) against r.

In order to test accuracy of this study, we employed the numerical values of the approximation scheme and the centrifugal term as shown in Tables 1, 2 and 3. Figures 10, 11 and 12 show a graph of centrifugal term and approximation scheme against the radial distance of the interacting particle. From the graph, it shows that the approximation scheme is good for the screening parameter $\alpha = 0.01$ meaning that the Unequal Scalar and Vectors Hulthen Potentials and Equal Scalar and Vector Modified Hellman plus Hylleraas Potential are short range potential.

Table 1. Variation of Square Inverse Radial Distance and Approximation Scheme with Radial Distance for $\alpha = 0.01$

R	$\frac{1}{r^2}$	$\frac{\alpha^2}{\left(1-e^{-\alpha r}\right)^2}, 4\alpha^2 \frac{e^{-2\alpha r}}{\left(1-e^{-\alpha r}\right)^2}$
0.2	25.000000	25.05004168
1.0	1.0000000	1.010041750
2.0	0.250000	0.255041833
3.0	0.111000	0.11448636
4.0	0.0625000	0.065042001
5.0	0.040000	0.042042084
6.0	0.027778	0.027486613
7.0	0.020408	0.021878987
8.0	0.012346	0.016917336
9.0	0.012346	0.013499210
10.0	0.010000	0.011042504

Table 2. Variation of Square Inverse Radial Distance and Approximation Scheme with Radial Distance for $\alpha = 0.1$

R	$\frac{1}{r^2}$	$\frac{\alpha^2}{\left(1-e^{-\alpha r}\right)^2}, 4\alpha^2 \frac{e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}$
1.0	1.0000000	1.104
2.0	0.250000	0.1525
3.0	0.111000	0.00341
4.0	0.0625000	0.001458
5.0	0.040000	0.000805
6.0	0.027778	0.0005098
7.0	0.020408	0.0003515

8.0	0.012346	0.0002569
9.0	0.012346	0.0001959
10.0	0.010000	0.00015

Table 3. Variation of Square Inverse Radial Distance and Approximation Scheme with Radial Distance for $\alpha = 1$

R	$\frac{1}{r^2}$	$\frac{\alpha^2}{\left(1-e^{-\alpha r}\right)^2}, 4\alpha^2 \frac{e^{-2\alpha r}}{\left(1-e^{-2\alpha r}\right)^2}$
1.0	1.0000000	2.5026
2.0	0.250000	1.3375
3.0	0.111000	1.1075
4.0	0.0625000	1.0377
5.0	0.040000	1.0136
6.0	0.027778	1.005
7.0	0.020408	1.0018
8.0	0.012346	1.00067
9.0	0.012346	1.00025
10.0	0.010000	1

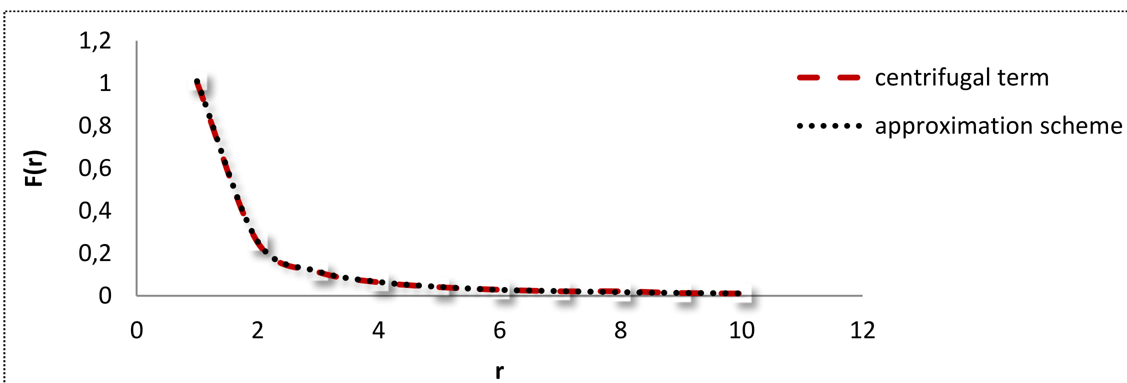

Figure 12. Comparison of The Centrifugal Term And The Approximation Scheme for $\alpha = 0.01$.

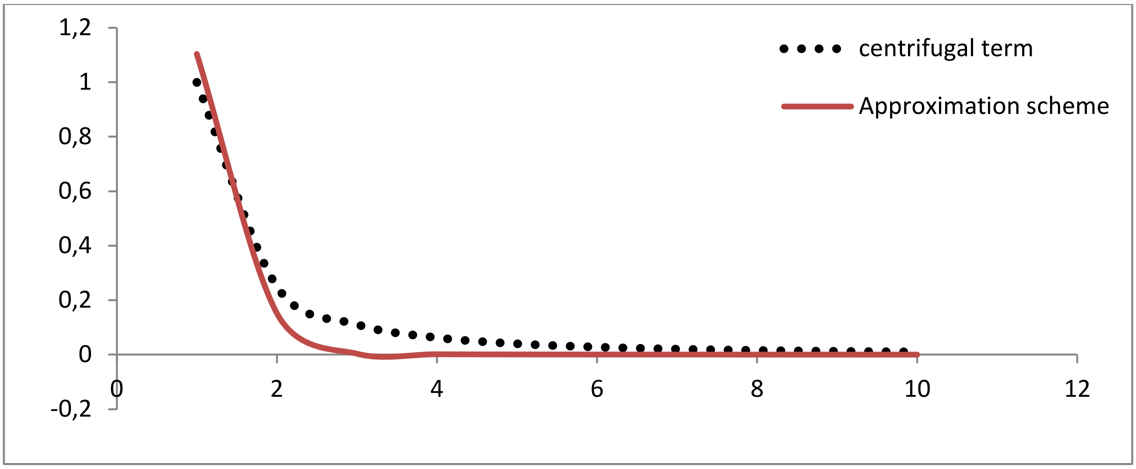

Figure 13. Comparison Of The Centrifugal Term And The Approximation Scheme for $\alpha = 0.1$.

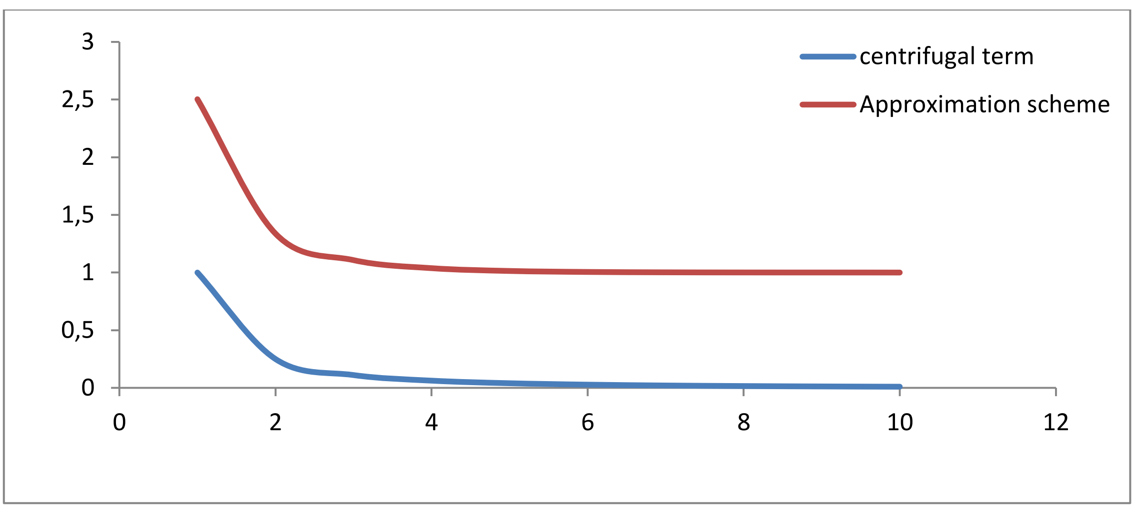

Figure 14. Comparison Of The Centrifugal Term And The Approximation Scheme for $\alpha = 1$

4. CONCLUSIONS

In this study, the approximate bound state solutions of Klein-Gordon equation Unequal Scalar and Vectors Hulthen Potentials and Equal Scalar and Vector Modified Hellman plus Hylleraas Potential via parametric Nikiforo-Uvarov (NU) method were obtained. The energy eigenvalues and the corresponding total normalized wave function in terms of the Jacobi polynomials were also obtained. The numerical energy eigenvalues obtained in this study is presented in Table 1 and 2. The behaviour of these potential were discussed in Figure 3 and 4. The results from the Unequal Scalar and Vectors Hulthen Potentials have many applications in the solid state physics, atomic physics, nuclear and particle physics and chemical physics.

The results from the Equal Scalar and Vector Modified Hellman plus Hylleraas Potential is used in the study of inner shell ionization problem, Electron core, Alkali-hydride molecule and in Solid state Physics (Amlan et al., 2013.)

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