Available online at www.worldscientificnews.com

World Scientific News

WSN 57 (2016) 276-287 EISSN 2392-2192

# Hamiltonian cycle containg selected sets of edges of a graph

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**ABSTRACT**

The aim of this paper is to characterize for every $k \geq 1$ all $(l + 3)$-connected graphs $G$ on $n \geq 3$ vertices satisfying $P(n + k)$: $\mathrm{d}_G(x,y) = 2 \;\Rightarrow\; \max\{\mathrm{d}(x,G),\mathrm{d}(y,G)\} \geq \frac{n+k}{2}$ for each pair of vertices $x$ and $y$ in $G$, such that there is a path system $S$ of length $k$ with $l$ internal vertices which components are paths of length at most 2 satisfying:

$$P:\; u_1u_2u_3 \subset S \text{ and } \mathrm{d}(u_1,G),\; \mathrm{d}(u_2,G) \geq \frac{n+k}{2} \;\Rightarrow\; \mathrm{d}(u_3,G) \geq \frac{n+k}{2},$$

such that $S$ is not contained in any hamiltonian cycle of $G$.

***Keywords*:** Cycle, hamiltonian cycle, matching, path

## 1. INTRODUCTION

We consider only finite graphs without loops and multiple edges. By V or V($G$) we denote the vertex set of graph $G$ and respectively by E or E($G$) the edge set of $G$. By d($x$,$G$) or d($x$) we denote *the degree of a vertex x in the graph G* and by d($x$,$y$) or d$G$($x$,$y$) *the distance between x and y in G.*

**Definition 1.1.** (cf [7]) *Let* $k, s_1, \ldots s_l$ *be positive integers. We call S a path system of length k if the connected components of S are paths:*

$$P^1: \quad x_0^1 x_1^1 \ldots x_{s_1}^1,$$
$$\vdots$$
$$P^l: \quad x_0^l x_1^l \ldots x_{s_l}^l$$

*And* $\sum_{i=1}^{l} s_i = k$.

Let $S$ be a path system of length $k$ and let $x \in V(S)$. We shall call $x$ *an internal vertex* if $x$ is an internal vertex (cf [2]) in one of the paths $P^1, \ldots, P^l$.

If $q$ denotes the number of internal vertices in a path system $S$ of length $k$ then $0 \leq q \leq k - 1$. If $q = 0$ then $S$ is *a k-matching* (i.e. a set of $k$ independent edges).

Let $G$ be a graph and let $S$ be a path system of length $k$ in $G$. Let paths $P^1: x_0^1 x_1^1 \ldots x_{s1}^1$, …, $P^l: x_0^l x_1^l \ldots x_{sl}^l$ be components of $S$. We can define a new graph $\tilde{G}$ and a matching $M_S$ in

$$M_S = \{xy : x = x_0^i, y = x_{s_i}^i, i = 1, \ldots, l\}$$

$$\mathrm{V}(\widetilde{G}) = \mathrm{V}(G) \setminus \bigcup_{i=1}^{l} \{x_1^i, \ldots, x_{s_i - 1}^i\}$$

$$\mathrm{E}(\widetilde{G}) = M_S \cup \{xy : x, y \in \mathrm{V}(\widetilde{G}) \text{ and } xy \in \mathrm{E}(G)\}$$

Let $H$ be a subgraph or a matching of $G$. By $G \backslash H$ we denote the graph obtained from $G$ by the deletion of the edges of $H$.

**Definition 1.2.** *F is an H-edge cut-set of G if and only if* $F \subset E(H)$ *and* $G \backslash F$ *is not connected.*

**Definition 1.3.** *F is said to be a minimal H-edge cut-set of G if and only if F is an H-edge cut-set of G which has no proper subset being an edge cut-set of G.*

**Definition 1.4.** (cf [5]) *Let G be a graph on* $n \geq 3$ *vertices and* $k \geq 0$. *G is k-edge-hamiltonian if for every path system P of length at most k there exists a hamiltonian cycle of G containing P.*

Let $G$ be a graph and $H \subset G$ a subgraph of $G$. For a vertex $x \in V(G)$ we define the set $N_H(x) = \{y \in V(H) : xy \in E(G)\}$. Let $H$ and $D$ be two subgraphs of $G$. $E(D,H) = \{xy \in E(G) : x \in V(D) \text{ and } y \in V(H)\}$. For a set of vertices $A$ of a graph $G$ we define the graph $G(A)$ as the subgraph induced in $G$ by $A$.

In the proof we will only use oriented cycles and paths. Let $C$ be a cycle and $x \in V(C)$, then $x^-$ is *the predecessor of* $x$ and $x^+$ is its *successor*. We denote the number of components of a graph $G$ by $\omega(G)$.

**Definition 1.5.** (cf [1]) *Let W be a property defined for all graphs of order n and let k be a non-negative integer. The property W is said to be k-stable if whenever G + xy has property W and* $\mathrm{d}(x,G) + \mathrm{d}(y,G) \geq k$ *then G itself has property W.*
J.A. Bondy and V. Chvátal [1] proved the following theorem, which we shall need in the proof of our main result:

**Theorem 1.1.** *Let n and k be positive integers with* $k \leq n - 3$*. Then the property of being k-edge-hamiltonian is* $(n + k)$*-stable.*

In 1960 O. Ore [6] proved the following:

**Theorem 1.2.** *Let G be a graph on* $n \geq 3$ *vertices. If for all nonadjacent vertices* $x,y \in \mathrm{V}(G)$ *we have*

$$\mathrm{d}(x,G) + \mathrm{d}(y,G) \geq n$$

*then G is hamiltonian.*

Geng-Hua Fan [3] has shown:

**Theorem 1.3.** *Let G be a 2-connected graph on* $n \geq 3$ *vertices. If G satisfies*

$$\mathrm{P}(n): \quad \mathrm{d}_G(x,y) = 2 \Rightarrow \max\{\mathrm{d}(x,G), \mathrm{d}(y,G)\} \geq \frac{n}{2}$$

*for each pair of vertices x and y in G, then G is hamiltonian.*

The condition for degree sum in Theorem 1.2 is called *an Ore condition* or *an Ore type condition for graph G* and the condition $\mathrm{P}(k)$ is called *a Fan condition* or *a Fan type condition for graph G.*

Later many *Fan type theorems* and *Ore type theorems* has been shown.

Now we shall present Las Vargnas [8] condition $\mathcal{L}_{n,s}$.

**Definition 1.6.** *Let G be graph on* $n \geq 2$ *vertices and let s be an integer such that* $0 \leq s \leq n$*. G satisfies Las Vargnas condition* $\mathcal{L}_{n,s}$ *if there is an arrangement* $x_1, \ldots, x_n$ *of vertices of G such that for all i,j if*

$$1 \leq i < j \leq n, \;\; i + j \geq n - s, \;\; x_i x_j \notin \mathrm{E}(G),$$

$$\mathrm{d}(x_i, G) \leq i + s \;\; \textit{and} \;\; \mathrm{d}(x_j, G) \leq j + s - 1$$

*then* $\mathrm{d}(x_i,G) + \mathrm{d}(x_j,G) \geq n + s$*.*

Las Vargnas [8] proved the following theorem:

**Theorem 1.4.** *Let G be a graph on* $n \geq 3$ *vertices and let* $0 \leq s \leq n - 1$*. If G satisfies* $\mathcal{L}_{n,s}$ *then G is s-edge hamiltonian.*

Note that condition $\mathcal{L}_{n,s}$ is weaker then Ore condition.

Later Skupień and Wojda proved that the condition $\mathcal{L}_{n,s}$ is sufficient for a graph to have a stronger property (for details see [7]). Wojda [9] proved the following Ore type theorem:

**Theorem 1.5** *Let G be a graph on $n \geq 3$ vertices, such that for every pair of nonadjacent vertices x and y*

$$\mathrm{d}(x,G) + \mathrm{d}(y,G) > \frac{4n-4}{3}.$$

*Then every matching of G lies in a hamiltonian cycle.*

In 1996 G. Gancarzewicz and A. P. Wojda [4] proved the following Fan type theorem:

**Theorem 1.6.** *Let G be a* 3*-connected graph of order $n \geq 3$ and let M be a k-matching in G. If G satisfies* $\mathrm{P}(n+k)$ :

$$\mathrm{d}(x,y) = 2 \;\Rightarrow\; \max\{\mathrm{d}(x),\mathrm{d}(y)\} \geq \frac{n+k}{2}$$

*for each pair of vertices x and y in G, then M lies in a hamiltonian cycle of G or G has a minimal odd M-edge cut-set.*

In this paper we shall find a Fan type condition under which every path system of length $k$ in a graph $G$ lies in a hamiltonian cycle.

For notation and terminology not defined above a good reference should be [2].

## 2. RESULT

**Theorem 2.1.** *Let G be a graph on $n \geq 3$ vertices and let S be a path system of length k with l internal vertices which components are paths of length at most 2 such that if $P:\ u_1u_2u_3 \subset S$ and* $\mathrm{d}(u_1,G), \mathrm{d}(u_2,G) \geq \frac{n+k}{2}$ *then* $\mathrm{d}(u_3,G) \geq \frac{n+k}{2}$ *. If G is* $(l+3)$*-connected and G satisfies* $\mathrm{P}(n+k)$ :

$$\mathrm{d}_G(x,y) = 2 \;\Rightarrow\; \max\{\mathrm{d}(x,G),\mathrm{d}(y,G)\} \geq \frac{n+k}{2}$$

*for each pair of vertices x and y in G, then S lies in a hamiltonian cycle of G or the graph $\widetilde{G}$ has a minimal odd $M_S$-edge cut-set.*

Note that under assumptions of Theorem 2.1 we have $0 \leq l \leq \lfloor \frac{k}{2} \rfloor$ .

It is clear that Theorem 1.6 is a simple consequence of Theorem 2.1.

## 3. PROOF

**Proof of Theorem 2.1.**

Consider $G$ and $S$ as in the assumptions of Theorem 2.1.

We can now define the set $A$:

$$A = \{x \in V(G) : \mathrm{d}(x, G) \geq \frac{n+k}{2}\}.$$

Note that if $x$ and $y$ are nonadjacent vertices of $A$ then the graph obtained from $G$ by the addition of the edge $xy$ also satisfies the assumptions of the theorem. Therefore and by Theorem 1.1 we may assume that:

$$xy \in E(G) \quad \text{for any} \quad x, y \in A \quad \text{and } x \neq y. \tag{3.1}$$

By (3.1) $A$ induces a complete subgraph $G(A)$ of the graph $G$. Let $GV \setminus A$ be a graph obtained from $G$ by deletion of vertices of the graph $G(A)$ (i.e. vertices from the set $A$).

Now consider a component $D$ of the graph $GV \setminus A$.

Suppose that there exist two nonadjacent vertices in $D$. Since $D$ is connected we have two vertices $x$ and $y$ in $D$ such that $\mathrm{d}G(x,y) = 2$ and by the assumption that $G$ satisfies $\mathrm{P}(n + k)$ we have $x \in A$ or $y \in A$, a contradiction.

So we can assume that every component of $GV \setminus A$ is a complete graph $K_\iota$, $\iota \in I$, joined with $G(A)$ by at least $l + 3$ edges.

If $K_{\iota 0}, K_{\iota 1} \in \{K_\iota\}_{\iota \in I}$ are such that $\iota_0 \neq \iota_1$ then:

$$N(K_{\iota_0}) \cap N(K_{\iota_1}) = \emptyset. \tag{3.2}$$

In fact, suppose that $N(K_{\iota 0}) \cap N(K_{\iota 1}) \neq \emptyset$. Then we have a vertex $y \in K_{\iota 0}$ and a vertex $y' \in K_{\iota 1}$ such that $\mathrm{d}G(y,y') = 2$ and by $\mathrm{P}(n + k)$ either $y \in A$ or $y' \in A$. This contradicts the fact that $K_{\iota 0}$ and $K_{\iota 1}$ are two connected components of $GV \setminus A$.

If $C \subset G$ is a cycle in $G$ then be $GV \setminus C$ we denote a graph obtained from $G$ by deletion of vertices of the cycle $C$.

The graph $G$ consists of a complete graph $GV \setminus A$ and of a family of complete components $\{K_\iota\}_{\iota \in I}$.

Let $K \in \{K_\iota\}_{\iota \in I}$.

Let $P : u_1u_2u_3$ be a path of length 2 from $S$. $P$ is called a $A$-ear if $u_1$, $u_3 \in A$ and $u_2 \in \mathrm{V}(K)$, and respectively a $K$-ear if $u_1$, $u_3 \in \mathrm{V}(K)$ and $u_2 \notin \mathrm{V}(K)$ (in this case $u_2 \in A$).

If $\mathrm{E}(K,A) \cap \mathrm{E}(S) \neq \emptyset$, then in $\mathrm{E}(K,A) \cap \mathrm{E}(S)$ we can have a family of ears and a number of edges from $\mathrm{E}(S)$ which does not form any ear.

Now we shall define a cycle $C$. First consider a path containing only all $A$-ears. Next we add to this path all remaining vertices from $A$ and all edges from the set $\mathrm{E}(S) \cap \mathrm{E}(G(A))$. All those edges and vertices form the cycle $C$.

Note that the cycle $C$ performs the following conditions:

- $C$ contains all edges of $\mathrm{E}(S) \cap \mathrm{E}(GV \setminus A)$ and all vertices of $A$. (3.3)
- If $K_{\iota 0}$ and $K_{\iota 1}$are two different components of $GV \setminus C$ then
  $N(K_{\iota 0}) \cap N(K_{\iota 1}) = \emptyset$. (3.4)
- Let $x \notin V(C)$, $y \in V(C)$ and $xy \in E(G)$ then: (3.5)
  if $y$ is not an internal vertex of $S$, then $y \in A$,
  if $y^-$ is not an internal vertex of $S$, then $y^- \in A$,

if $y^+$ is not an internal vertex of $S$, then $y^+ \in A$.

Such cycle exists since $GV \setminus A$ is a complete graph and $G$ satisfies (3.2).

Hence $G$ is $(l + 3)$-connected, every component of $GV \setminus C$ is a complete graph joined with $C$ by at least 3 edges which ends are not internal vertices of $S$.

Let $K$ be a connected component of $GV \setminus C$.

We shall show that we can extend the cycle $C$ over all vertices of $K$, over all edges of $S$ in $K$ and over all edges of $S$ joining $K$ with $C$ preserving the properties (3.3) — (3.5) or that the graph $\widetilde{G}$ has a minimal odd $M_S$-edge cut-set.

**Case 1**

Among the edges joining $K$ with $C$ there are no edges from path system $S$. Since $G$ is $(l + 3)$-connected $K$ is joined with $C$ by at least 3 edges which ends are not internal vertices of $S$. We have $x_iy_i$ such that $x_i \in K, y_i \in C$ and $y_i^-, y_i, y_i^+ \in A$, for $i = 1,2,3$. We can assume that vertices $x_1$, $x_2$, are joined by one path $P$ from $S$. Here $P$ is directed from $x_2$ to $x_1$.

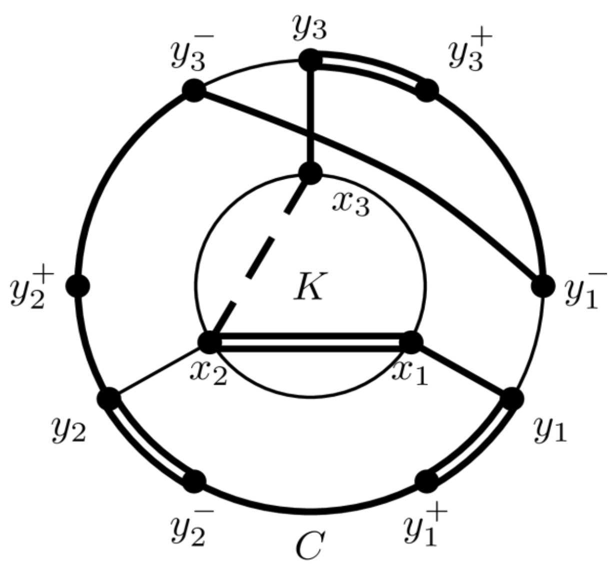


Figure (3.1).

Suppose that also $y_1y_1^+$, $y_2y_2^- \in E(S)$. If $y_3y_3^+ \in E(S)$ (then $y_3$ is a start vertex of one path from $S$ directed towards $x_1$) we can consider the cycle (see Figure (3.1)):

$$C' : \quad y_3x_3v_1 \ldots v_kPy_1y_1^+ \ldots y_2^- y_2y_2^+ \ldots y_3^- y_1^- \ldots y_3^+, \tag{3.6}$$

where $v_1 \ldots v_k$ is a path containing all remaining vertices from the set

$$V(K) \setminus (P \cup \{x_3\})$$

and edges from the set $(E(S) \cap E(K)) \setminus E(P)$.

when $y_3^- y_3 \in E(S)$ we can carry out similar construction of cycle $C'$.

$$C': \quad y_3x_3v_1 \ldots v_kPy_1y_1^+ \ldots y_2^- y_2y_1^- \ldots y_3^+ y_2^+ \ldots y_3^- .$$

Note that we can do the same if $y_1$ and $y_2$ joined by one path from $S$.

If $y_2$ and $y_3$ are end vertices of the same path from $S$ or $y_2y_2^+$, $y_3y_3^- \in E(S)$ we can carry out a similar construction.

Suppose that $y_1y_1^+ \in E(S)$. We can consider the cycle:

$$C': \quad y_3x_3v_1 \ldots v_kPy_1y_1^+ \ldots y_2^- y_2y_1^- \ldots y_3^+ y_2 \ldots y_3^- .$$

Supposing that $y_1^- y_1$, $y_3^- y_3 \in E(S)$ a good extension of $C$ should be the cycle:

$$C': \quad y_3x_3v_1 \ldots v_kPy_1y_1^- \ldots y_3^- y_1^+ \ldots y_2^- y_2 \ldots y_3^- .$$

The last two cycles are good also if $y_2$ and $y_3$ are end vertices of the same path from $S$. when $y_iy_1^+ \in E(S)$ for $i = 1,\ldots,3$ we can define $C'$ as in (3.6).

It is clear that the new cycle $C'$ fulfills (3.3) — (3.5) and is an extension of $C$ such that

$$\mathrm{V}(C) \subset \mathrm{V}(C') \quad \text{and} \quad ((\mathrm{E}(C) \cup \mathrm{E}(K)) \cap \mathrm{S}),\ (\mathrm{E}(C,K) \cap \mathrm{S}) \subset \mathrm{E}(C'). \qquad (3.7)$$

If among the edges joining $K$ with $C$ there are no edges from path system $S$ then all other situations can be reduced to those presented above.

**Case 2**

Among the edges joining $K$ with $C$ there are some edges from path system $S$.

Since $G$ is $(l + 3)$-connected $K$ may be joined with $C$ by a family of $K$-ears and at list three edges which ends are not internal vertices of $S$.

Hence $G$ and $S$ satisfies the following condition: if $P: u_1u_2u_3 \subset S$ and $d(u_1,G)$, $d(u_2,G) \geq \frac{n+k}{2}$ then $d(u_3,G) \geq \frac{n+k}{2}$ edges from $E(C,K) \cap E(S)$ may be as on Figure (3.2).

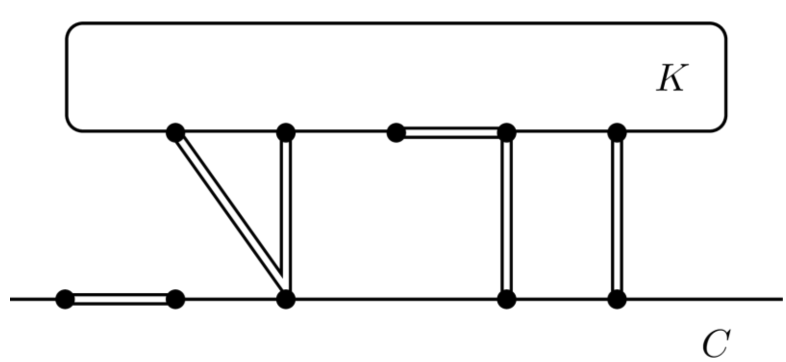


Figure (3.2).

The graph $\widetilde{G}$ has a minimal odd $M_S$-edge cut-set if there is a component $K$ of $GV \setminus C$ which is joined with cycle $C$ only by an odd number of edges from E($S$) or an odd number of edges from E($S$) and edges from E($G$) \ E($S$) with at least one end vertex in the set of internal vertices of path system $S$. In those cases the theorem is proved, so we may assume that $\widetilde{G}$ has no minimal odd $M_S$-edge cut-set

**Subcase 2.1.**

Among edges joining $K$ with $C$ we have an even number of edges from E($S$), say $s = 2r$, ($r \geq 1$) which does not form any ear.

So we have vertices $x_1,\dots,x_{2r} \in K$ and $y_1,\dots,y_{2r} \in C$ such that $x_iy_i \in$ E($S$), for $i = 1,\dots,2r$. We can assume that each edge $x_iy_i$ is in path of length 2 from path system $S$. Then we have vertices $x_i^+ \in$ V($K$) such that $x_i^+x_i \in$ E($S$), for $i = 1,\dots,2r$.

Let $u, v \in$ V($C$) be such that all edges from E($C$,$K$) ∩ E($S$) lying between $u$ and $v$ belong to some ears. In the cycle $C$ we have a path $W: uc_1\dots c_kv \subset C$. We shall define a new path $Q(u,v)$. If $u$ and $v$ are not in any ear. The path $Q(u,v)$ is a path joining $u$ with $v$ such that E($W$) ∩ E($S$) ⊂ E($Q(u,v)$) and $Q(u,v)$ contains all $c_i$ such that $c_i$ is not an internal vertex of a $K$-ear. In other words $Q(u,v)$ arises from $W$ by removing internal vertices of all $K$-ears. It is possible because if $c_i$ is an internal vertex of a $K$-ear then $c_i^-$, $c_i^+ \in A$.

When $u$ is internal vertex of a $K$-ear, then we start the path $Q(u,v)$ from the first vertex $c_i$ which is not internal vertex of any $K$-ear. If $v$ is internal vertex of a $K$-ear, then we end the path $Q(u,v)$ from the last vertex $c_i$ which is not internal vertex of any $K$-ear.

The construction of $Q(u,v)$ is shown on figures (3.3) — (3.5).

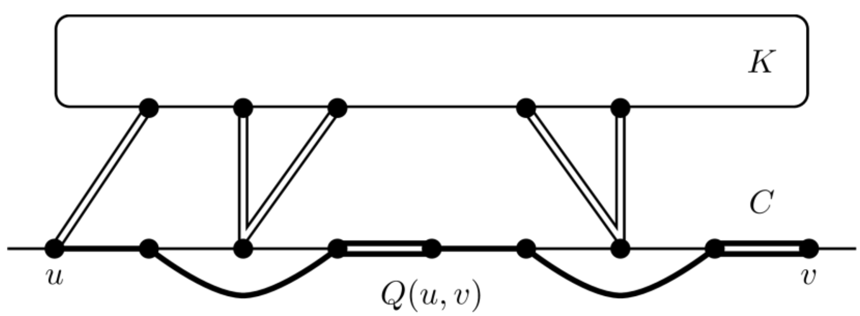


Figure (3.3).

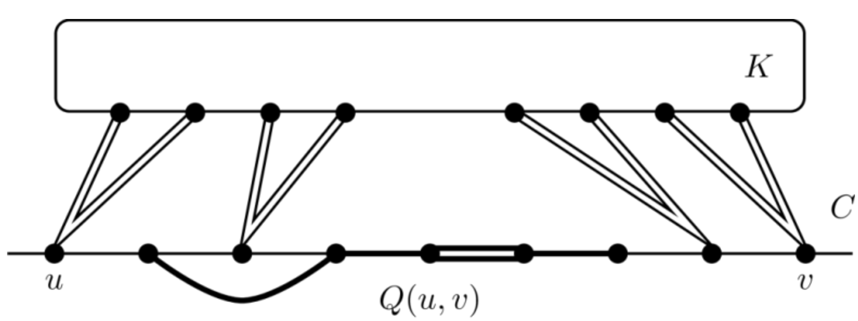


Figure (3.4).

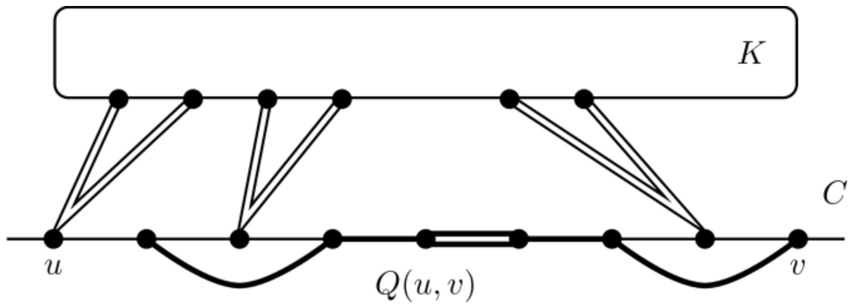


Figure (3.5).

First consider the path $P_K$ containing only all $K$-ears. Now we can define the extension of the cycle $C$ as follows (see Figure (3.6) (for $r = 2$)):

$$C' : \quad \begin{array}{l} y_1 x_1 x_1^+ P_K x_2^+ x_2 Q(y_2, y_1^+) Q(y_2^+, y_3) x_3 x_3^+ \ldots y_{2r-1} \\ x_{2r-1} x_{2r-1}^+ v_1 \ldots v_s x_{2r}^+ x_{2r} Q(y_{2r}, y_{2r-1}^+) Q(y_{2r}^+, y_1), \end{array}$$

where $x_{2r-1}^+ v_1 \ldots v_s x_{2r}^+$ is a path containing all remaining vertices of $K$ and edges of $E(S) \cap E(K)$, this path exists because $K$ is complete.

It is clear that the new cycle $C'$ fulfils (3.3) — (3.5) and (3.7).

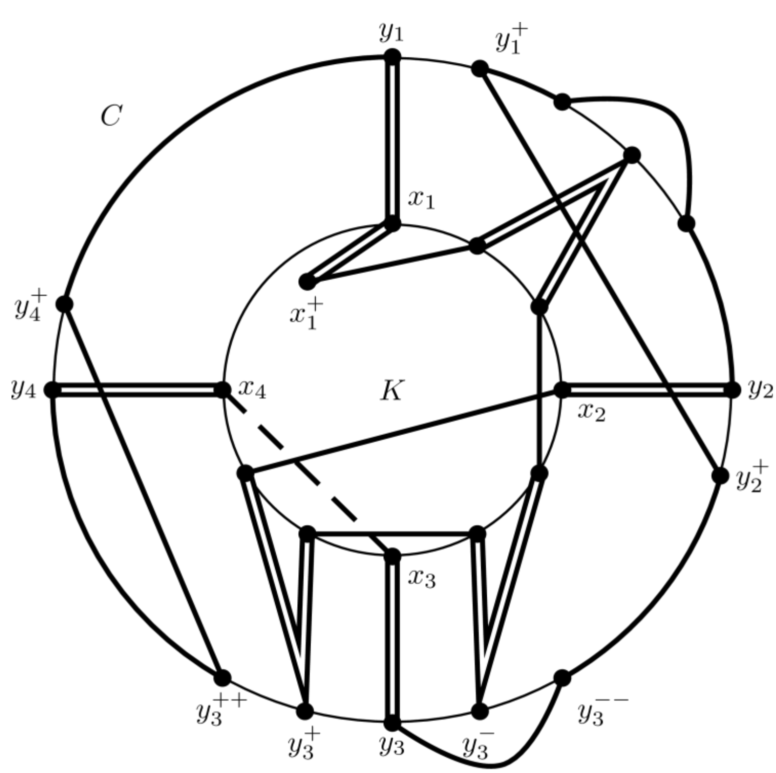


Figure (3.6).

**Subcase 2.2.**

Among edges joining $K$ with $C$ we have an odd number of edges from E($S$), say $s = 2r - 1$, ($r \geq 1$) which does not form any ear.

So we have vertices $x_1,\dots,x_{2r-1} \in$ V($K$) and $y_1,\dots,y_{2r-1} \in$ V($C$) such that $x_iy_i \in$ E($S$). We can assume that each edge $x_iy_i$ is in path of length 2 from path system $S$. Then we have vertices $x_i^+ \in$ V($K$) such that $x_i^+x_i \in$ E($S$), for $i = 1,\dots,2r - 1$.

Since we have assumed that $\widetilde{G}$ has no minimal odd $M_S$-edge cut-set we have at least one edge say $xy$, ($x \in K$, $y \in C$) such that $xy \notin$ E($S$), $x$ and $y$ are not an internal vertices of $S$.

We shall consider four subcases according as $x$ or $y$ are extremities of an edge from the set E($S$).

Suppose that $y \notin \{y_1,\dots,y_{2r-1}\}$ and $x \notin \{x_1,\dots,x_{2r-1}\}$. In this case we have a vertex $y_{i0} \in V$ ($C$), ($i_0 \in \{1,\dots,2r - 1\}$) such that on the cycle $C$ the vertices are ordered as follows: $y_{i0}\dots y\dots y_{i0+1}$.

Consider a path $xv_1\dots v_sx_{i0+1}^+x_{i0+1}$ containing all vertices from the set

V($K$) $\setminus\{x_1,x_1^+,\dots,x_{i0},x_{i0}^+,x_{i0+2},x_{i0+2}^+,\dots,x_{2r-1},x_{2r-1}^+\}$ all $K$-ears and all edges from E($S$) $\cap$ E($K$).

If $y^-y \in$ E($S$) consider the following cycle $C'$ :

$$C' : \quad \begin{aligned} &y^-yxv_1\dots v_sx^+_{i_0+1}x_{i_0+1}Q(y_{i_0+1},y^+)Q(y^+_{i_0+1},y_{i_0+2})x_{i_0+2}x^+_{i_0+2}x^+_{i_0+3}x \\ &Q(y_{i_0+3},y^+_{i_0+2})Q(y^+_{i_0+3},y_{i_0+4})\dots Q(y_{i_0},y^+_{i_0-1})Q(y^+_{i_0},y^-), \end{aligned}$$

satisfying properties: (3.2) — (3.5) and (3.7).

when $r = 1$ the edge $xy$ must be independent with all $x_iy_i$, so now we have $r \geq 2$.

Suppose that for $y \notin \{y_1,\dots,y_{2r-1}\}$ and there is an $i_0 \in \{1,\dots,2r - 1\}$ such that $x = x_{i0}$. In this case $x_{i0}x_{i0}^+ \notin$ E($S$).

If $yy^- \in$ E($S$) then we define a new cycle $\tilde{C}$ as follows:

$$\tilde{C} : \quad y^-yx_{i_0}Q(y_{i_0},y^+)Q(y^+_{i_0},y^-).$$

and consider the complete graph $D$ obtained from $K$ by deletion of the vertex $x_{i0}$.

$D$ is a component of $G_{V\setminus\tilde{C}}$. Note that $\tilde{C}$ and $D$ satisfies conditions (3.3) — (3.5) and (3.7). Since $r \geq 2$ $D$ is joined with $\tilde{C}$ by an even number of edges from E($S$), which does not form any ear and then we can proceed as in subcase (2.1).

Suppose that for some $i_0,j_0 \in \{1,\dots,2r - 1\}$ $x = x_{i0}$, $y = y_{j0}$, and ($i_0 \neq j_0$).

First consider the case $r = 2$ and vertices $y_1$, $y_2$, $y_3$ are ordered in $C$ as follows: $y_1\dots y_2\dots y_3$.

We can assume that $y = y_1$, $x = x_3$ ($x_3x_3^+ \notin$ E($S$)) and then consider the cycle:

$$C' : \quad y_3x_3y_1x_1v_1\dots v_sx_2Q(y_2,y_1^+)Q(y_1^-,y_3^+)Q(y_2^+,y_3),$$

where $x_1v_1\dots v_sx_2$ is a path containing all remaining vertices from $K$ all $K$-ears and all edges from E($S$) $\cap$ E($K$).

Again the cycle $C'$ has properties: (3.2) — (3.5) and (3.7).

when $r > 2$ we have $y_l x_l \in E(S)$ and we assume that in the cycle $C$ vertices are ordered as follows: $y_{j0} \ldots y_l \ldots y_{i0}$. Now we can define a new cycle $\tilde{C}$:

$$C' : \quad y_{i_0} x_{i_0} y_{j_0} x_{j_0} x_l^+ x_l Q(y_l, y_{j_0}^+) Q(y_{j_0}^-, y_{i_0}^+) Q(y_l^+, y_{i_0}).$$

and consider the complete graph $D$ obtained from $K$ by deletion of the vertices $x_{i0}$, $x_l$ and $x_{j0}$.

$D$ is a component of $G_{V \setminus \tilde{C}}$. Note that $\tilde{C}$ and $D$ satisfies conditions (3.3) — (3.5) and (3.7). Since $r > 2$ $D$ is joined with $\tilde{C}$ by an even number of edges from E($S$), which does not form any ear and a family of ears, so we can proceed as in subcase (2.1).

**Subcase 2.3.**

Among edges from E($S$) joining $K$ with $C$ we have only edges which are forming $K$-ears.

Hence $G$ is $l + 3$ connected we have also at least 3 edges from E($G$) \ E($S$) which ends are not internal vertices of $S$.

This case is similar to the case 1. The only difference is fact that we have $K$-ears, but using paths $Q(u,v)$ we can extend the cycle as in case 1.

In all cases we have extended the cycle $C$, so the proof is complete.

## 4. CONCLUSIONS

The proof of Theorem 2.1 is an example of application of the closure technique. Note that the construction of the cycle $C$ in the closure of the graph $G$ is algorithmic but unfortunately it is possible that the cycle is using edges that does not belong to the initial graph.

Our result is an extension of Theorem 1.6.

## References

[1] J. A. Bondy and V. Chvátal, A method in graph theory, *Discrete Math.* 15 (1976) 111-135.

[2] J. A. Bondy and U.S.R. Murty, Graph theory with applications, MacMillan Press LTD, 1976.

[3] G. Fan, New sufficient conditions for cycles in graphs, *J. Combin. Theory Ser. B* 37 (1984) 221-227.

[4] G. Gancarzewicz and A. P. Wojda, Graphs with every k-matching in a hamiltonian cycle, *Discrete Math.* 213(1-3) (2000) 141-151.

[5] H. V. Kronk, Variations of a theorem of pósa, *in* The Many Facets of Graph Theory, ed. G. Chartrand and S.F. Kapoor, *Lect. Notes Math.* 110 (1969) 193-197.

[6] O. Ore, Note on hamiltonian circuits, *Amer. Math. Monthly* 67 (1960) 55.

[7] Z. Skupień and A. P. Wojda, On highly hamiltonian graphs, *Bull. Acad. Polon. Sci., Sér. Sci. Math. Astronom. Phys.* 22 (1974) 463-471.

[8] M. Las Vargnas, Sur une propriété des arbres maximaux dans un graphe, *C. R. Axad. Sci. Paris, Sér.* A 272 (1971) 1297-1300.

[9] A. P. Wojda, Hamiltonian cycles through matchings, *Demonstratio Mathematica* XXI(2) (1983) 547-553.

( Received 28 September 2016; accepted 12 October 2016 )