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On Unconventional Division by Zero

Jakub Czajko

Science/Mathematics Education Department, Southern University and A&M College, Baton Rouge, LA 70813, USA

E-mail address: sunswing77@netscape.net

ABSTRACT

Unconventional division by zero can apply to those cases in which some directly incomparable (i.e. unlike or represented in different algebraic bases) abstract objects taken from two distinct algebraic spaces are acted upon or need to be compared. Multispatial reality paradigm is thus necessary for overcoming conceptual inconsistencies arising in the domain of complex numbers when certain valid abstract reasonings from the hypercomplex domains of quaternions and octonions are carried over back to the 2D complex domain.

***Keywords*:** Unconventional division by zero, multispatial algebraic multiplication

1. INTRODUCTION

Ancient Indian scientist Brahmagupta considered numbers as abstract entities, but from the premise that $1\times0=0$ from which one would be inclined to derive $(0/0)=1$ because any entity divided by itself should equal unity but he concluded that $(0/0)=0$ instead [1], which may be called alternative [2] or unconventional [3] division by zero . Although the premise is operationally wrong [2] and conceptually questionable [4], it is still adhered to, even today. But the unconventional division by zero $(0/0)=0$ is not always properly understood, presumably because of subsequent conclusions to the contrary. For another Indian scientist Bhaskara II showed later that inverse zero equals infinity by the intuitively clear method of exhaustion [1], which Euler also used and explained in [5], much, much later, though.

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Numbers were traditionally considered as discrete entities [6] yet identified with their values. Ingenious inverse mapping is briefly discussed in Banach spaces [7] and inverse monodromy problem and Riemann-Hilbert factorization was extensively discussed in [8]. Banach algebra and point at infinity is concisely discussed in [9] and the behavior of trajectories at infinity was fairly well discussed in terms of Poincaré sphere in [10].

Yet Jan Dieudonné successfully used the unconventional division by zero (0/0)=0 too in modern abstract mathematical context [11]. More recently Saburou Saitoh and several other scientists apparently resuscitated the ancient Indian idea of unconventional division by zero and exemplified its usefulness in [12-15]. Yet in order to attain a mathematical synthesis of the two apparently contradictory ideas of division by zero, both of which – if implemented – could offer unrestricted operation of algebraic division by zero, would require multispatial approach to the abstract mathematical (and thus also to physical) reality.

The need for having an unambiguous unrestricted division by zero was also motivated by the need to synchronize spaces with algebraic operations on quasigeometric spacetime [16] and to associate all spacetime points with numbers sets without placing any artificial restrictions on abstract algebraic operations [2,3]. The other reason arises from the formerly unexpected mathematical fact that existence of 4D spacetime (ST) implies presence of a (dual to the ST) quasispatial temporal structure of 4D timespace (TS) [17] that is overlaid over the ST [18]. Though not widely advertised, this research is not dormant. Yet it was restricted to topology [19] and algebras, both of which rely on the set-theoretical approach to mathematics that ignored the synthetic need to match structures to procedures.

In fact, Clifford algebra of spacetime actually pertains to the timespace [20] p.28, i.e. to the 4D temporal structure that lays over the quasispatial structure of the 4D spacetime. The two quasispatial structures of ST and TS, may thus be viewed as two distinct views on the same underlying 4D quasispatial structure in its spatial and temporal form, respectively.

Nevertheless, one could still ask why is the division of zero by zero – as it was proposed in [2] and [3] – important? The first reason for the importance is that division by zero is actually multiplication by infinity and so it was implemented in [2], [3], [16]. The other reason has much deeper roots. Although these roots were not always seen as indispensable, they cause incomparability of objects represented in incompatible algebraic bases.

2. INCOMPATIBLE ALGEBRAIC REPRESENTATIONS

It is known that despite its usefulness in mathematics and physical sciences, the domain of complex numbers is operationally incomplete. This fact compelled Hamilton to invent quaternions. Yet certain conceptual ideas developed for the more operationally complete hypercomplex domains of quaternions and octonions, for which those ideas work very well indeed, are not backward compatible when applied to the 2D complex domain.

When one constructs antiautomorphic octonion conjugate map represented in terms of left and right action on 8D octonions $\mathbb{O}$: $X^* = \frac{1}{6}\sum_{n=0}^{7} -e_n X e_n$ the common coefficient of the summation is $\frac{1}{6} = \frac{1}{8-2}$ [20] p.22. The conjugate map for the 4D quaternions $\mathbb{H}$ has structurally (i.e. formwise) similar coefficient, namely: $\frac{1}{2} = \frac{1}{4-2}$ [20] p.15. Yet if we extend this construction back to the complex case, the analogous conjugate map for 2D complex numbers ℂ demands inverse zero as the common coefficient of the same form, namely: $\frac{1}{0} = \frac{1}{2-2}$ which leads to the – traditionally evaded – or just avoided evaluation:

$$-\frac{1}{2-2}[-1(\mathrm{x}+\mathrm{iy})1 - \mathrm{i}(\mathrm{x}+\mathrm{iy})\mathrm{i}] = \frac{0}{0} \tag{1}$$

whose result obviously demands by necessity a certain division of zero by zero [20] p.22.

Dr. Geoffrey M. Dixon very honestly admitted explicitly that "in [his] experience $\frac{0}{0}$ can equal pretty much anything (as a limit), so it's a little hard to tell if this is x-iy, or not. It could be a balloon or perhaps a goat, neither of which are in general equal (x+iy)*=x-iy" [20] p.22. I am not kidding – just quoting him verbatim.

One can see that mere defining does not necessarily create the (usually either explicitly or tacitly defined) conjugate relationship, because definitions are just existential postulates in disguise [21], [22]. This latter statement does not mean that the conjugate relationship could not exist, but that perhaps there is also something else that enables its existence. In the sense definitions and postulates can create either nonexistent reality or – as it is in the latter case – can inadvertently conceal existence of an intermediate auxiliary enabling reality.

Note that the conventional complex-analytic stereographic projection that maps sphere into complex plane is not conceptually uncontested mapping. For even though every sphere has two poles, only one of them is mapped to the – traditionally assumed as single-valued number – zero, the other (opposite) pole is mapped to set-valued infinity. Euler pointed out that division by zero is doable if zero and infinity are reciprocal [5] p.23, [4] p.435.

By contradiction thus, if traditional mathematics does not know how to implement the division of zero by zero it is because we cannot house zero and infinity within the same complex domain/space. This conclusion requires that infinity – as entity reciprocal to zero – should dwell in a distinct space rather than in the very same space in which zero is housed. Complex numbers in geometry were discussed already in [23] and in [24].

Hence both the conventional as well as the unconventional division by zero can make operational sense in a multispatial reality, first for an intraspatial division and the latter for an interspatial division, respectively.

It is obvious that for the mapping (1) to be unambiguous either the regular zero should become multi-valued or infinity should become single-valued, or perhaps both should become multivalued, if mathematics wants to avoid confusion. That is why I proposed two separate yet paired spaces: one for housing the regular zero and the other for the infinity that is reciprocal to zero, to reside in. Notice that if the regular zero is depicted in a real 3D space $\mathbb{R}^3$ whose algebraic basis is denominated in meters [m] then the dual reciprocal 3D space in which the infinity – as inverse of zero – can dwell must have an algebraic reciprocal native basis denominated in spatial frequency [1/m] i.e. inverse meters, for if the two spaces were merged together it could create confusion when someone would just add magnitude denominated in meters to yet another one denominated in inverse meters. Such an alleged "space" could not become unambiguous metric space and it can only be posing as a set generated by some selection rules. Yet the terms 'set' and 'space' are frequently used interchangeably. Paired Banach spaces and related issues were discussed in [25].

Nobody in her right mind would do that, but an irresponsible mapping placing both types of magnitude within the same algebraic space without realizing that such a placement requires heterogeneous basis that virtually permits that mistake. It was rather unlikely to see in traditional mathematical treatises algebraic bases treated in the same way as vectorial bases are handled. But if an extra basis should be added just to prevent misidentification of magnitudes then it is the same as adding an extra space or quasispatial structure with each of these spaces retaining its own homogeneous algebraic basis that is native to the given space.

For it is easier to maintain two distinct spaces, each of which is equipped with its native single homogeneous basis, than to alter the operational rules just in order to accommodate operations on objects immersed within spaces having a heterogeneous algebraic basis. However, the latter case is necessary for handling 4D spatial and quasispatial structures, which are neither really single spaces nor truly simple spaces as some traditionally-minded mathematicians evasively used to stipulate. Vectorial bases were discussed in [26], [27].

In other words: by "rigorously" defining a simple space and then dumping the qualifier 'simple' (allegedly for the sake of "simplicity") one does not solve any related issues but merely sweeps them under the proverbial carpet where the conceptually unresolved issues can linger for centuries and thus can stink theoretically. That is irony if not deceit.

In summary: The idea of complex conjugate is forward compatible when extended to the aforesaid hypercomplex numbers. But some features of conjugates that are acquired during transition to (and thus work for) the hypercomplex numbers, when applied to the lower-dimensional (LD) complex domain, are not always backward compatible with features of those higher-dimensional (HD) conjugates. While complex domain is somewhat incomplete in comparison to the two hypercomplex domains, the algebraic operational incompleteness can also imply presence of a structural gap, for from a (new) synthetic point of view, to each operational procedure there should exist a (corresponding to it) constructible geometric or quasigeometric structure and vice versa. Also, presence of artificial restrictions – such as the unwarranted former prohibition on division by zero – hinted at possible existence of some procedural incompleteness, in addition to the hypothesized structural incompleteness.

3. INTRASPATIAL DIVISION BY ZERO

In his discussion of the alleged impossibility of division by zero Euler wrote that the word infinite signifies the operational fact that actually one can never arrive at a definite end of the series of the fractions, for the denominator must be infinte or infinitely great in order that the fraction may be reduced to zero, which he identified with nothing [5] p.23.

In order to avoid operational pitfalls, however, we must recognize that numbers matched to geometric points should be treated as algebraic entities having some definite algebraic and spatial attributes as well. In the sense the numeric value of the entity called zero is null, but it is not really nothing. Euler surely did his best, but when numbers are treated as algebraic entities on equal footing with geometric points then some of Euler's conclusions may appear conceptually unacceptable. Division by zero is thus not impossible anymore.

For algebraic division of zero by zero can be implemented as the (inverse to division) operation of multiplication of numerator by the (dual reciprocal/inverse to zero) infinity

$$\frac{0}{0} = 0 \cdot \infty = 1 \quad if \quad \frac{0_p}{0_p} = 1 \tag{2}$$

[16], which is an intraspatial operation if it is deployed within the same primary algebraic space P that is equipped with a homogeneous native algebraic basis p, after making

conversion (if necessary) of magnitudes from a foreign basis to the native algebraic basis of the space in which the algebraic operation is to be performed [2], [3].

The conversion is necessary only when the zero resides in primary spaces equipped with either different native bases, or one of the zeros resides within a dual reciprocal space associated/paired with the given primary space P, of course. Abstract algebraic reciprocity was discussed in terms of algebraic specializations in [28] and abstract duality was ingeniously discussed in [29], [30], [31], and in the set-theoretic setting in [32], [33], and in terms of Poincaré duality [34], and in terms of dual groups [35], or as topological duality [36], and in dual notions of theory of categories [37].

The validity of eq. (2) is contingent upon the condition that the native algebraic basis p of the primary algebraic space P is homogeneous. While the regular scalar multiplication sign for purely algebraic entities is omitted by convention, the dot scalar multiplication operates on abstract algebraic entities in a multispatial context, where it plays the role analogous to that of the regular scalar multiplication of vectors that is rendered by bold dot.

The right-hand side (RHS) of the eq. (2) can be one condition for the SSR paradigm

$$\frac{0_p}{0_p} = 1 = 0 \cdot \infty \tag{3}$$

because only in the same single space the representation of zero with respect to the given heterogeneous algebraic basis p of a certain primary algebraic space P is equal to any other formal representation of zero cast in the same conceptual setting, provided each subsequent representation of zero is obtained by using proper/legitimate operational means.

The eq. (3) is based on the Eulerian approach to infinity explained in [5], [16]. It was conceived under the former SSR paradigm. The problem is that the SSR paradigm is only accidentally true when everything is directly representable in a single space and is cast in (or converted into) the same algebraic or geometric basis common to all objects involved.

4. TRANSIT FROM INTRASPATIAL TO INTERSPATIAL CASE

The tautology on the LHS of the following eq. (4) in conjunction with the eq. (3) implies

$$\frac{0_p}{0_p} = 1 \Leftrightarrow 0_p \cdot \infty_p = 0_q \cdot \infty_q \Rightarrow 1_p = 1_q \Leftrightarrow \frac{0_p}{0_q} = \frac{\infty_q}{\infty_p} \tag{4}$$

for the spaces P and Q are just two distinct views of the same object. If so, then the middle equation implies that multiplicative neutral units in P and Q should be equal – as having definite value that remains unchanged.

Under the SSR paradigm in the real domain and in an algebraic setting the siding of multiplicands is not important for commutative operations. But under the MSR paradigm assuming left-handed multiplications by the foreign basis index just as it is common with vectors in the inner product of vectors of <bra | ket> [38] is essential. The multiplication order is important because pure complex and hypercomplex numbers (i.e. the geometric parts of these numbers) can be identified with algebraic vectors.

From the RHS of the eq. (4) we can obtain, on one hand, the following expressions:

$$\frac{0_p}{0_q} = \frac{\infty_q}{\infty_p} = \frac{1}{\infty_p \cdot 1/\infty_q} = \frac{1}{\infty_p \cdot 0_q} = \frac{1}{1_{pq}} \tag{5}$$

and, on the other hand, we get from the RHS of the eq. (4) and the RHS of the eq. (5):

$$\frac{0_p}{0_q} = \frac{0_p}{1/\infty_q} = \infty_q \cdot 0_p = 1_{qp} = \frac{1}{1_{pq}} \tag{6}$$

where the two-index subscripts denote transitions from space identified by the index basis in the first/left position to the index basis in the second/right position.

Note that the equations shown in this section are just obvious tautologies, independent of the character of the respective algebraic bases p,q, or of nature of the spaces P,Q, that are denominated/determined by the corresponding algebraic bases.

From the eq. (6) we obtain the following equations

$$\frac{0_p}{0_q} = 1_{qp} = \frac{1_{qp}}{1_q} = \frac{1}{1_{pq}} \tag{7}$$

which are derived in Appendix A. The eq. (7) can be further extrapolated to

$$\frac{0_p}{0_q} = 1_{qp} = \frac{1_{qp}}{1_q} \quad \Rightarrow \quad \frac{1}{1_{pq}} = \frac{1_p}{1_{pq}} \tag{8}$$

given the fact that the multiplicative neutral unit does not change during transitions between algebraic bases because it should have definite single value.

Furthermore, from the eq. (6) we can derive the following abstract relationships that are valid under the MSR paradigm and represented here in two-index notation:

$$\infty_q \cdot 0_p = 1_{qp} = \frac{1}{1_{pq}} \Rightarrow 1_{qp} \cdot 1_{pq} = 1 \quad \Rightarrow \quad \infty_q \Mapsto 1_{qp} \;\; \& \;\; 0_p \Mapsto 1_{pq} \tag{9}$$

which shows that while infinity in two-index notation determines the upward direction of transition (i.e. from the foreign to the native algebraic basis – hence from the secondary to the primary space), zero in two-index notation determines downward transition (i.e. from the native to the foreign algebraic basis – hence from the primary to the secondary space).

The secondary space can be either the dual reciprocal space associated with the primary space or yet another primary space yet equipped with quite different algebraic basis, which may not be compatible with the native basis of the given/first primary space.

That the relations (9) resemble inversions is also pretty clear even under the former unspoken SSR paradigm, where the mappings: $0 \rightarrow 1$, $1 \rightarrow 0$, $\infty \rightarrow \infty$ refer to abstract inversions [39] p.39, which clearly change a cross ratio into its conjugate [39] p.44, where these issues were briefly discussed too. Compare also the classic complex inversion transformation function w=1/z that was concisely discussed in [40].

However, under the MSR paradigm we maintain sets of paired spaces and thus at least two distinct zeros (one in each space) and consequently also two distinct infinities if we choose two pairings. Double pairings of spatial structures shall be discussed elsewhere.

From the eq. (9) we can devise the upward transition between two spaces as follows:

$$\frac{0_p}{0_q} = 1_{qp} \quad \Longrightarrow \quad \frac{0^p}{0_q} \uparrow \; \Longleftarrow \quad \infty_q \Longmapsto 1_{qp} \tag{10}$$

which can be uncontroversial because the mapping of set-valued infinity into single-valued entity resulting from the division of zero by zero on the LHS is of many-to-one character.

The RHS of the eq, (9) also portends the unconventional division by zero for from

$$0_p = 1_{pq} \tag{11}$$

it would not be possible to index the zero with both bases of the two distinct spaces P, Q:

$$\frac{0_p}{0_q} = 0_p = |0| \quad \Longrightarrow \quad \frac{0^p}{0_q} \downarrow \; \Longleftarrow \quad 0_p \Longmapsto 1_{pq} \tag{12}$$

highlighted in red. What may appear controversial under the unwarranted SSR paradigm is not only imaginable but obviously feasible in nontraditional operational terms. Therefore, the mapping representing the unconventional division by zero is highlighted here in red.

The eq. (12) is thus an abstract representation of operational implementation of the downward transition. It may be provocative from the standpoint of the SSR paradigm not because it is invalid but because it attempts to map single-valued entity into many-valued or set-valued one without specifying how to split the single value into possibly infinite set of multiple values, which would normally require designing an abstract algorithm for the implementation of quite unambiguous, yet also quite distinct, one-to-many mapping.

Although such a splitting algorithm can be devised for finite spread of values, the spread could be infinite. Yet one could easily assign single value to many values if we spread the multiple values in a distinct space even if it contains infinite continuum of numbers/points.

It might be tempting to equate 0_{pq} with 1_{pq} if it were possible, but it is obviously not

$$0_{pq} \neq 1_{pq} \tag{13}$$

which would not only be logically inconsistent, but it would try to equate set-valued zero (or infinity) with single-valued multiplicative neutral unit during the same many-to-one transit.

Yet this failure does not imply that the eq. (12) is invalid, but merely that its LHS means that the algebraic space must be somehow folded. The paired spaces P and Q are virtually folded into a single hyperspace with heterogenous basis comprising the two bases. For if folded, the two spaces virtually permit the operation resulting in unconventional division by zero. This demonstrates the heuristic power of the new mathematical synthesis and truly astonishing effectiveness of pairing of spatial or quasispatial structures.

Nonetheless, it is possible to imagine an alternative interspatial division by zero that can also become operational implementation of the unconventional division by zero.

5. UNCONVENTIONAL DIVISION BY ZERO AS INVERSION

Two-index notations can encapsulate more possibilities of relationships than single-index representations, depending on character of the spaces involved and on nature of the algebraic bases of these spaces, for any given space can hold the same geometric objects each of which is cast in different algebraic basis. Appendix B shows by example how a single-index depiction can be expanded to the corresponding to it two-index representation.

Indexing the neutral units 0,1, with two algebraic bases of the primary spaces I concur that the inversive mappings 0 → 1, 1 → 0 [39] can reflect their correspondence within a multispatial hyperspace under the MSR paradigm. This correspondence can be evaluated as:

$$\frac{0_p}{0_q} = 0_{pq} \Longmapsto \frac{0_p}{0_q} = \frac{0_p}{1_p} \cdot \frac{1_p}{0_q} = \frac{1}{\infty_p} \cdot \infty_{qp} = \frac{\infty_{qp}}{\infty_p} \Longrightarrow \frac{\infty_{qp}}{\infty_{pq}} = \frac{1}{\frac{\infty_{pq}}{\infty_{qp}}} = \frac{1}{\infty_{pq} \cdot 0_{qp}} = \frac{1}{1_{pq}} = 1_{qp} \tag{14}$$

which is more precise rendition than the single-index expression of the very same mapping. Yet the two-index rendition doubles the possibilities of mappings. For the eq. (14) implies

$$0_{pq} \Longmapsto 1_{qp} \tag{15}$$

and by abstract duality I can also conjecture that the apparently additive reverse (i.e. not the multiplicative inverse) relationship is conceptually quite analogous to the one shown above

$$0_{qp} \Longmapsto 1_{pq} \tag{16}$$

which supports abstract interspatial inversive relations in the multispatial hyperspace and so suggest asymmetric allocation of neutral elements within paired primary spaces. Notice that while the symbol $\Longrightarrow$ denotes logical/material implication, the symbol $\Longmapsto$ implies mapping. For it would not be appropriate, in general, to interpret it as identical with just relating.

Notice the similarity between the above mappings and negatives in logic, whose laws appear as if the form of multiplicative law in logic: A∧¬A=0 resembles that of algebraic addition and the form of additive law in logic: A∨¬A=0 resembles that of algebraic multiplication [41]. Here ¬A denotes negation of a sentence A and 0,1 mean false and true, respectively. The logic of incomplete truth and incomplete knowledge is discussed in [42].

6. UNCONVENTIONAL DIVISION BY ZERO YIELDS NULL

However, the – fairly easy to conceive – intraspatial operation of division by zero and the corresponding to it reciprocal multiplication hint at the possibility of devising also an operationally feasible interspatial multiplication and division operations – compare also [3].

As I mentioned it above, the eq. (2): $if\ \frac{0_p}{0_p} = \frac{0}{0} = 0 \cdot \infty = 1$ and (3): $\frac{0_p}{0_p} = 1 = 0 \cdot \infty$ reflect the conditions for the former SSR paradigm which tacitly assumed that the abstract realm of mathematics can be equated with single space. The formal expansion from the SSR paradigm to the MSR paradigm can also be stipulated by the above relations (15) and (16).

Hence the tentative operational condition for the MSR paradigm can also be offered as

$$if\ p{\neq}q\ \ then\ \Longrightarrow\ \ \frac{0_p}{0_q} = [NULL] \coloneqq |0|\ \ \Longrightarrow\ \ \frac{0_p}{0_q} = |0| \qquad (17)$$

where the representations of zeros in different homogeneous native algebraic bases p and q of certain primary spaces P and Q respectively, are not directly comparable, and thus their ratio is null, which may be equated to zero value. Thus p≠q is the preliminary condition of the SSR paradigm. It shall be expanded and then perhaps reformulated when needed.

The eq. (17) is thus more formal rendition of the unconventional division of zero by zero where the null value means formal incomparability of representations of the distinct zeros. Notice similarity of the RHS of the eq. (17) and the unconventional division by zero, which is not inadmissible if it is qualified by the algebraic bases wherein its formal – as opposed to actual – validity can be ascertained. Hence the unconventional division by zero is not necessarily valid under the SSR paradigm, but it seems as a necessary alternative when zeros represented within different bases p≠q are not directly comparable. Folding or otherwise distorting the primary or their associated dual reciprocal spaces can also impose the condition shown in the eq. (17).

Bases are useful for finite dimensional vector spaces but do not always have nice generalizations to infinite dimensions [43]. In the multispatial hyperspace each of its spaces is finite dimensional but there is no upper limit on the number of its spatial components.

Under the MSR paradigm, however, bases are of importance for both algebraic and vector spaces, because dimensionality of single spaces is restricted by orthogonality to 3D and dimensionality of quasispatial structures is capped at 4D by conveniently forgotten or misinterpreted results of Abel and Galois [17], [18]. Hence the MSR paradigm virtually posts no limit on dimensionality, which is quantized and spread among spatial 3-tuples or in quasispatial 4-tuples among the quasispatial structures from which each multispatial hyperspace is composed. Nevertheless, in order to ensure that the spaces and quasispatial structures are constructible, the synthetic approach to mathematics requires to match every geometric structure constructed under the MSR paradigm, whether it is only algebraic or analytic or differential, to an operational procedure. In a sense thus, the MSR paradigm replaces the – essentially unstructured (but then acquiring a structure imposed on it by often arbitrary definitions), which then tacitly play the role of concealed existential postulates – former single-space reality with an abstract mathematical realm structured by the operations that can be performed on objects immersed in the realm.

The eq. (1) demonstrates thus failure of the SSR paradigm not only in handling of higher dimensions (HD) but also dilemmas resurfacing in lower dimensional (LD) cases when confronted with reasonings devised for objects represented in HD, which seem too obvious to mention them when predecessors of the HD objects were originally defined in LD, yet too abstract to comply with. The obvious failure reinforces the need to control the geometric buildups of spatial structures by operational procedures, for no matter how sophisticated our mathematical definitions may appear to be, their static nature seems incapable of predicting the prospective dynamic character of our intended creations. Often instead of attempted sophistication the former traditional mathematics ends up with sophistry, as in the eq. (1).

7. APPLICATION OF INTERSPATIAL DIVISIONS BY ZERO

Traditional ways of handling infinite intervals and then improper integrals involving infinities in the real domain is well exemplified in [44].

Complex analysis relies mainly on successful treatment of infinities as a kind of pest to be avoided or circumvented by series. The complex domain $\mathbb{C}$ shall be elaborated elsewhere. The methods developed in both the real and complex analysis can give acceptable numerical evaluations, but they tend to create conceptual confusion and even predictive disasters when infinities cannot be evaded or circumvented. But under the MSR paradigm many of the infinities are treatable in a very similar way the nonsingular problems are analyzed. Stable Lagrangian singularities are listed in [45]. Review of singularities research is found in [46].

Based on the eq. (17) the original eq. (1) can now be extended in the complex domain as

$$-\frac{1}{2-2}[-1(\mathrm{x}+\mathrm{iy})1-\mathrm{i}(\mathrm{x}+\mathrm{iy})\mathrm{i}]=\frac{0}{0}=0 \tag{18}$$

which cannot be equated with conventional division by zero: $\frac{0}{0}=1$ for the term: $\frac{1}{2-2}=\frac{1}{0}$ belongs to a space of integer dimensions (i.e. it is denominated by the algebraic basis whose numbers/elements signify countable set of integer numbers representing spatial dimensions: 1,2,3,…) and the remaining part of the eq. (18): $[-1(\mathrm{x}+\mathrm{iy})1-\mathrm{i}(\mathrm{x}+\mathrm{iy})\mathrm{i}]=0$ belongs to a complex space that is regarded as continuum, unless it is assumed to be a discrete lattice. The complex space cannot be identified with the set of discrete integer dimensions. For dimensions are just enumerated as labels depicted by numerals in form of discrete integers.

In the sense, the two zeros above are not really comparable, unless the complex space would also be denominated by the same basis of countable integer dimensions. But this is unlikely for the traditional pure mathematics did not really care about specifying algebraic bases of any set-theoretical number spaces. Recall that the eq. (17) accidentally works for the hypercomplex numbers because in hypercomplex domains it does not reach the barrier at which the conversion of zero to the basis, in which infinity is represented, is imperative.

The eq. (18) is an example of the unconventional division by zero that is necessary when expressions cast in different algebraic bases are either to be acted upon or compared. I am not faulting the author of the equation. But we must not suppress the developments that can contradict some old ones, even if the new ideas falsify some previously proven theorems.

One can see that the improper expressions or equations could lead to confusion even though at first glance they look like mathematically legitimate expressions. The fact that some mathematical expressions can be written down does not necessarily mean that they are valid, not to mention true. It is not good enough to specify sets without their selection rules or spaces without their bases (whether algebraic or vectorial/geometric) or mappings without their transition rules or even functions without their admissible transformations.

The new/recent synthetic approach prefers specification of the underlying paradigms as well. The scope of validity of proofs is thus limited by the scope of validity of the paradigms upon which the proofs relied, regardless of our recognition of presence of the paradigms in the terms underlying the axiomatics employed by the proofs. Thus, no eternal proofs exist.

Constructive proofs are often used to prove existence theorems [47]. Existential proofs can create nonsenses. Proofs by contraposition are discussed in [48], [49] and combinatorial proofs are examplified in [50]. Theorems of negative character are discussed in [51].

In the sense traditional mathematics is conceptually deficient even if it is not incorrect, because even rigorously proven theorems could be invalidated under new paradigms. Proofs can only confirm validity of derivations from the axioms and primitive notions that have been accepted as selfevident on the ground of (often unmentioned) questionable paradigms.

8. APPENDIX A

The above eq. (7) can be formally derived as follows:

$$\frac{0_p}{0_q} = \frac{0_p}{1_p} \cdot \frac{1_p}{0_q} = \frac{0_p}{1_p} \cdot \frac{1_p}{\frac{1_q}{\infty_q}} = \frac{0_p}{1_p} \cdot \frac{\infty_q}{1_q} \cdot 1_p = \frac{\infty_q}{1_q} \cdot \frac{0_p}{1_p} \cdot 1_p = \frac{1_{qp}}{1_q}$$

and since the multiplicative neutral unit remains unchanged during transitions between algebraic bases because it has always definite single value, the above eq. (8) can then be derived as follows:

$$\frac{0_p}{0_q} = \infty_q \cdot 0_p = 1_{qp} = \frac{1}{1_{pq}} = \frac{1_p}{1_{pq}}.$$

9. APPENDIX B

In order to see the operational differences between the single-index representation and the two-index representations of the same algebraic objects immersed within unspecified spaces belonging to a multispatial structure of a hyperspace, let us consider the following.

If the old tautology $0_p = 0_p \cdot 1$ is accepted at its face value and then just extrapolated as

$$0_p = 0_p \cdot (1) = 0_p \cdot (\infty_q \cdot 0_p) = 0_p \cdot 1_q \qquad (19)$$

then when we use zeros represented in two distinct bases p and q native to two different primary spaces P and Q, respectively, we could write very similar chain of expressions as

$$\frac{0_p}{0_q} = 0_q \quad or \quad \frac{0_p}{0_q} = \frac{0_p}{1_p} \cdot \frac{1_p}{0_q} = 0_p \cdot \infty_q = \frac{\infty_q}{\infty_p} = \frac{1}{\frac{\infty_p}{\infty_q}} = \frac{1}{\infty_q \cdot 0_p} = \frac{1}{1_q} = 1_p \qquad (20)$$

from which I conclude that zero in one primary space corresponds to the neutral unit 1 in yet another dual primary space of the given multispatial structure to which both spaces belong

$$0_q \rightrightarrows 1_p \qquad (21)$$

so that the eqs. (19) and (20) can apparently give two distinct and quite different evaluations of the additive neutral element that is usually identified with algebraic zero, for we obtain:

$$\frac{0_p}{0_p} = 1_p \quad or \quad \frac{0_p}{0_q} = 0_q \tag{22}$$

depending on whether the native representation is homogeneous or heterogeneous. Note that in two-index renditions the spaces P, Q, can be different and thus can hint at possible presence of heterogeneous bases appearing on a deeper conceptual level of handling the multispatial structure comprising these spaces. One can see that if the zeros are represented in the same basis then the conventional division by zero is induced, but if the zeros are formally represented in distinct bases then the unconventional division by zero is induced.

The eq. (22) indicates the need to use algebraic bases explicitly and consequently also two-index notation for bases. Thus – contrary to pure-mathematical tradition – if one avoids specifying the (allegedly superfluous) algebraic bases then formal representations are either underdefined or underdetermined. Both cases are clearly detrimental for abstract reasonings.

10. CONCLUSIONS

The conventional division by zero $\frac{0_p}{0_p} = 1_p$ does apply when both zeros are represented in the same algebraic basis p of the algebraic or geometric space P, for in this case both zeros and all other numbers represented in the same basis are directly comparable, provided the basis p is homogeneous. If one of the zeros belongs to yet another space, then its representation should be converted from the native (or foreign) basis of the other space to the basis p of the space P within which the operation of division is to be performed.

It has also been shown that the unconventional division by zero $\frac{0_p}{0_q} = 0_q$ is applicable in those cases where the algebraic representations of objects operated on are not converted to the same algebraic basis in which their representations can be comparable. Such cases can also happen when the bases of the spaces in which the zeros/objects are immersed are either incompatible (and thus directly incomparable) or perhaps misrepresented in discordant algebraic or geometric bases.

Note that although null and zero may have the same algebraic value assigned, their conceptual meanings are definitely different. The unconventional division by zero actually yields null, which means incompatibility, not zero, which would be confusing if it were admissible. Yet we used to assign zero as the arithmetic/algebraic value of the entity called null, which is just a shorthand for incompatibility. While zero resides in nonempty sets or spaces, the null/nothing resides nowhere, i.e. null dwells in an empty set.

Since traditional mathematics routinely disregarded algebraic bases and thus virtually allowed for inconsistent reasonings to emerge, the unconventional division by zero must not be precluded as inherently invalid but could be treated as a herald of the necessity to deploy more sophisticated mathematics than the traditionally developed one. For it has been demonstrated above, that the unconventional division by zero arose quite naturally from otherwise valid reasonings though performed in conceptually underdefined or perhaps underdetermined setting.

The unconventional division by zero can thus be used for probing incompatibility of the spatial (or quasispatial) algebraic (or operational, in general) representations of abstract algebraic procedures operating/acting on objects immersed in a multispatial hyperspace composed of various spatial and/or quasispatial structures.

The possibility of coexistence of the conventional and unconventional divisions by zero implies that algebraic fields of numbers with unrestricted algebraic operations (i.e. without prohibition on division by zero) could be unambiguously matched to physical fields composed of spatial or perhaps abstract quasispatial structures comprising geometric or quasigeometric points, the latter being spacetime points, for instance.

The fact that traditional mathematics instituted prohibition on division by zero rather than honestly admitting that perhaps they were unable to conceive and/or implement the idea of unrestricted algebraic operations (including division by zero) does not mean that their – contemptible, as it often is with respect to misunderstood alternative conceptual ideas – handling of abstract mathematics is the only valid approach to doing mathematics. Only unanticipated curious experimental results or unbiased theoretical/thought experiments – not prohibitions or other arbitrary decrees – should guide the development of pure mathematics.

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