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# Strong Efficient Co-Bondage Number of Some Graphs

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**ABSTRACT**

Let G = (V, E) be a simple graph. A subset S of V(G) is called a strong (weak) efficient dominating set of G if for every v ∈ V(G), $|N_s[v] \cap S|=1$. ($|N_w[v] \cap S|=1$), where $N_s(v)$ = {u ∈V(G) : uv∈ E(G), deg u ≥ deg v}. ($N_w(v)$ = {u ∈V(G) : uv∈ E(G), deg v ≥ deg u}). The minimum cardinality of a strong (weak) efficient dominating set of G is called the strong (weak) efficient domination number of G and is denoted by $\gamma_{se}$(G) ($\gamma_{we}$(G)). A graph G is strong efficient if there exists a strong efficient dominating set of G. The strong efficient co-bondage number $bc_{se}$(G) is the maximum cardinality of all sets of edges X ⊆ E such that $\gamma_{se}(G+X) \leq \gamma_{se}$(G). In this paper, the strong efficient co-bondage number of some standard graphs and some special graphs are determined.

***Keywords*:** domination, strong efficient domination, strong efficient co-bondage number

**AMS Subject Classification (2010):** 05C69

## 1. INTRODUCTION

Throughout this paper finite, undirected graphs without loops or multiple edges are considered. Let G = (V, E) be a simple graph. The concept of domination in graphs was introduced by Ore. A book on Domination theory in graphs was published by Acharya, Waliker and Sampathkumar. Two volumes on domination have been published by Haynes, Hedetniemi

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and Slater. Bange, Barkauskas and Slater defined an efficient dominating set of a graph. The concept of strong domination in graphs was introduced by Sampathkumar and Pushpalatha. The concept of strong efficient domination in graphs was introduced by N. Meena et al. The concept of co-bondage number was introduced in V.R. Kulli and B. Janakiram. Motivated by these definitions, the authors introduce the concept of strong efficient co-bondage number. In this paper, strong efficient co-bondage numbers of some standard graphs and some special graphs are studied.

## 2. PRELIMINARIES

Before proving the result some basic definitions, results and theorems are given.

**Definition 1.1:** A subset S of V(G) is called a strong dominating set of G if for every v ∈ V – S, there exists u ∈ S such that u and v are adjacent and deg u ≥deg v.

**Definition 1.2:** Let G = (V, E) be a simple graph. A subset S of V(G) is called a strong(weak) efficient dominating set of G if for every v ∈ V(G) ($|N_w[v]$ ∩ S|=1) where $N_s(v)$ = {u ∈ V(G) : uv∈ E(G), deg u ≥deg v} and $N_s[v] = N_s(v)$ ∪ {v}($N_w(v)$ = {u ∈ V(G) : uv∈ E(G), deg v≥ deg u} and $N_w[v] = N_w(v)$ ∪ {v}).

**Remark 1.3:** The minimum cardinality of a strong (weak) efficient dominating set of G is called the strong (weak) efficient domination number of G and is denoted by $\gamma_{se}$(G)($\gamma_{we}$(G)). A graph G is strong efficient if there exists a strong efficient dominating set of G. A graph which is not strong efficient is called a non-strong efficient graph.

**Theorem 1.4:** For any path $P_m$, $\gamma_{se}(P_m) = \begin{cases} n \ if \ m = 3n, \ n \in N \\ n+1 \ if \ m = 3n+1, \ n \in N \\ n+2 \ if \ m = 3n+2, \ n \in N \end{cases}$

**Theorem 1.5:** $\gamma_{se}(C_{3n})$ = n for all n ∈ $N$.

**Observation 1.6:** $\gamma_{se}$(G) = 1 if and only if G has a full degree vertex.

**Definition1.7:** A regular spanning sub graph of degree 1 is called 1-factor (1F).

**Theorem 1.8:** $K_{n,n}$−1F is strong efficient and $\gamma_{se}$($K_{n,n}$−1F) = 2, ∀n∈N.

**Theorem 1.9:** Let G= $D_{r,s}$, r, s ≥ 1 and r≤ s. $\gamma_{se}$(G) = r + 1.

**Definition 1.10:** The Coconut tree graph T(n,m) is obtained by joining the central vertex of the star $K_{1,m}$ and a pendent vertex of a path $P_n$ by an edge.

**Definition 1.11:** The Twig graph $G_n$ is obtained from the path $P_n$ by attaching exactly two pendent edges to each internal vertex of the path.

## 3. RESULT

**Definition 2.1:** The strong efficient co-bondage number $bc_{se}$(G) is the maximum cardinality of all sets of edges X ⊆ E such that $\gamma_{se}(G+X) \leq \gamma_{se}$(G).

**Illustration 2.2:** Consider the following graphs G and G + e.
In G, $\{v_1, v_5\}$ is the unique strong efficient dominating set. Hence $\gamma_{se}$(G) = 2. Let e = $v_1 v_5$. In G + e, $v_1$ is the full degree vertex. Therefore $\gamma_{se}$(G + e) = 1. Hence $bc_{se}$(G) = 1.

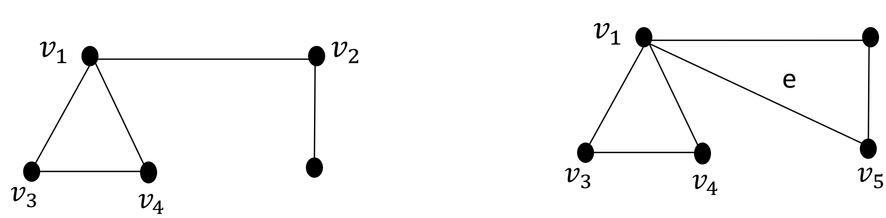


**Figure 2.1.**

**Theorem 2.3**

$$bc_{se}(P_m) = \begin{cases} 1 & if\ m = 3n+1\ or\ 3n+2, n \geq 1 \\ 3 & if\ m = 3n, n \geq 2 \end{cases}$$

**Proof:**

**Case (1):** Let G=$P_m$ , m = 3n+1, n ≥ 1. Let V(G) = $\{v_1, v_2, \dots, v_{3n}, v_{3n+1}\}$. Then $\{v_1, v_3, v_6, \dots, v_{3n}\}$ and $\{v_2, v_5, v_8, \dots, v_{3n+1}\}$ are two strong efficient dominating sets of G and $\gamma_{se}$(G) = n+1. Let e=$v_1 v_3$. Let H = G + e. Then S= $\{v_3, v_6, v_9, \dots, v_{3n}\}$ is the unique $\gamma_{se}$- set of H and |S| = n<$\gamma_{se}$(G). Hence $bc_{se}$(G) = 1.

**Case (2):** Let G =$P_m$ , m = 3n+2, n ≥ 1. Let V(G) = $\{v_1, v_2, v_3, \dots, v_{3n}, v_{3n+1}, v_{3n+2}\}$. Then $\{v_1, v_3, v_6, \dots, v_{3n}, v_{3n+2}\}$ is the unique strong efficient dominating set of G and $\gamma_{se}$(G) = n+2. Let e =$v_1 v_3$. Let H = G + e. Then S = $\{v_3, v_6, v_9, \dots, v_{3n}, v_{3n+2}\}$ is the unique $\gamma_{se}$- set of H and |S| = n+1 <$\gamma_{se}$(G). Hence $bc_{se}$(G) = 1.

**Case (3):** Let G =$P_m$ , m = 3n, n ≥ 2. Suppose one edge is added with G. Let H = G + e. Let S be a $\gamma_{se}$- set of H.

**Sub case (3a):** Let e=$v_1 v_{3n}$. Then G + e =$C_{3n}$ *and* $\gamma_{se}$(G + e) = n.

**Sub case (3b):** Let e=$v_i v_j$, i = 1 or j =3n. Let i = 1.

**Sub subcase (3b1):** Let e = $v_1 v_j$ such that d($v_1, v_j$) = 3k – 1, k ≥ 1. Then deg$v_j$ = Δ(H) and $v_j$ is the unique maximum degree vertex. Hence $v_j \in$ S. Therefore the vertices

$v_j, v_{j+3}, v_{j+6}, \ldots, v_{3n-3}$ belong to S.If $v_{3n-1} \in$ S, then $|N_s[v_{3n-2}] \cap S| = |\{v_{3n-3}, v_{3n-1}\}| = 2 > 1$, a contradiction. If $v_{3n-1} \notin$ S, then no vertex in S to strongly efficiently dominate $v_1$, a contradiction. Hence this sub-sub-case (3b1) does not exist.

**Sub sub case (3b2):** Let e = $v_1 v_j$ such that d($v_1, v_j$) = 3k, k ≥ 1. Since deg$v_j$ = Δ(H), then $v_j \in$ S. Hence $v_{j-3}, v_{j-6}, \ldots, v_4$ belong to S. If $v_2 \in$ S, then $|N_s[v_3] \cap S| = |\{v_2, v_4\}| = 2 > 1$, a contradiction. If $v_2 \notin$ S, then no vertex in S to strongly efficiently dominate $v_2$, a contradiction. Hence this sub subcase (3b2) does not exist.

**Sub subcase (3b3):** Let e = $v_1 v_j$ such that d($v_1, v_j$) = 3k+1, k ≥ 1. Proceeding as in sub-sub-case(3b2), $v_j, v_{j-3}, v_{j-6}, \ldots, v_5$ belong to S.If $v_2 \in$ S, then $| N_s[v_1] \cap S| = |\{v_2, v_j\}| = 2 > 1$, a contradiction. If $v_2 \notin$ S, then no vertex in S to strongly efficiently dominate $v_2$and $v_3$, acontradiction. Hence this subsubcase (3b3) does not exist. The proof is similar if j = 3n.

**Sub case (3c):** Let e = $v_i v_j$, 2 ≤ i , j ≤ 3n – 1.

**Sub subcase (3c1):** Let d $(v_i, v_j)$ =3k, k ≥ 1.Since deg$v_i$ = deg $v_j$ = Δ (H), then either $v_i$ *or* $v_j \in$ S**.** Suppose $v_i \in$ S**.** Let T = $\{v_i, v_{i+1}, \ldots, v_j\}$. Then the subgraph induced by vertices of T is$C_{3k+1}$ , k ≥ 1, which has no strong efficient dominating set. Hence H has no strong efficient dominating set, a contradiction. Therefore this subsubcase (3c1) does not exist. The proof is similar if $v_j \in$ S**.**

**Sub subcase (3c2):** Let d $(v_i, v_j)$ =3k – 1, k ≥ 1. Then $T_1 = \{v_2, v_5, \ldots, v_{3n-1}\}$ and $T_2 = \{v_1, v_4, v_7, v_{10}, \ldots, v_{3n-2}, v_{3n}\}$ are strong efficient dominating sets of H.$|T_1|$ =n and $|T_2|$ = n+1.Therefore$\gamma_{se}(H)$ = n. Hence$bc_{se}(G) \geq 2$.

**Sub subcase (3c3):** Let d$(v_i, v_j)$ =3k+1, k ≥ 1. Then S = $\{v_2, v_5, v_8, \ldots, v_{3n-1}\}$ is the unique strong efficient dominating set of H and $\gamma_{se}(H)$ = n. From these cases, we get $bc_{se}(G) \geq 2$.

**Sub case (3d):** Let the two edges be added with G. Let H = G + $\{e_1, e_2\}$.

**Sub subcase (3d1):** Let $e_1 = v_i v_j$ and $e_2 = v_i v_{j+1}$, i = 1, 2 ≤ $j$ ≤3n – 1. Then T = $\{v_2, v_5, v_8, \ldots, v_{3n-1}\}$ is a strong efficient dominating set of H. Further no set with k < n vertices is a strong efficient dominating set of G. $\gamma_{se}(H) \geq$ n. Hence $bc_{se}(G) \geq 3$.

**Sub subcase (3d2):** Let $e_1 = v_i v_j$ and $e_2 = v_i v_{j+1}$, 2 ≤ $i$ ≤ 3n – 2, i ≠ 3k+1 and 4 ≤ $j$ ≤3n – 1, k ≥ 1. Then $v_i$ is the unique maximum degree vertex and hence$v_i \in$S. Suppose i = 3k+1, k≥1. Then $v_{i-3}, v_{i-6}, v_4$ belong to S. As in subsubcase (3d2), H has no strong efficient dominating set, a contradiction. This subsubcase (3d2) does not exist. Suppose i ≠ 3k+1. Let T = N[$v_i$]. Then the subgraph induced by V – T is the path $P_{3n-5} = P_{(3n-6)+1} = P_{3(n-2)+1}$ and $\gamma_{se}(P_{3(n-2)+1})$ = n – 1. So$\gamma_{se}(H)$ = n.Hence $bc_{se}(G) \geq 3$.

**Subsubcase (3d3):** Let $e_1 = v_1 v_3$ and $e_2 = v_3 v_5$. Here $v_3$ strongly efficiently dominate$v_i$, 1 ≤ i ≤ 5. Then the subgraph induced by remaining vertices is $P_{3n-5}$. Therefore

$\{v_3, v_7, v_{10}, \ldots, v_{3n-2}, v_{3n}\}$ is the unique $\gamma_{se}$- set of H and hence $\gamma_{se}(H)$ = 2+n – 2 = n = $\gamma_{se}(G)$. Hence $bc_{se}(G) \geq 3$.

**Subsubcase (3d4):** Let $e_1 = v_i v_{i-2}$ and $e_2 = v_i v_{i+2}$, $4 \leq i \leq$ 3n– 3. Let T = $\{v_{i-2}, v_{i-1}, v_i, v_{i+1}, v_{i+2}\}$. Then the subgraph induced by the vertices of V – T is $P_r \cup P_s$ and $v_i$ belongs to any $\gamma_{se}$- set of H.

**Subsubsubcase (3d4a):** Let r = 3k and s = 3n – 3k – 5, k ≥ 1. Then $\gamma_{se}(H) = \gamma_{se}(P_r) + \gamma_{se}(P_s)$ + 1 =k+n – k – 2+1+1= n =$\gamma_{se}(G)$ .

**Subsubsubcase (3d4b):** Let r = 3k+1 and s = 3n – 3k – 6, k ≥ 0. Then $\gamma_{se}(H) = \gamma_{se}(P_r) + \gamma_{se}(P_s)$ + 1 = k+1+n – k – 2+1= n =$\gamma_{se}(G)$ .

**Subsubsubcase (3d4c):** Let r = 3k – 1 and s = 3n – 3k – 4, k ≥ 1. Then $\{v_2, v_5, v_8, \ldots, v_{3n-1}\}$ is the unique strong efficient dominating set of H. Hence$\gamma_{se}(H)$ = n =$\gamma_{se}(G)$. From the above all subsubsubcases, $bc_{se}(G) \geq 3$.

**Subsubcase (3d5):** Let $e_1 = v_i v_j$ and $e_2 = v_r v_s$ such that d$(v_i, v_j)$ = d$(v_i, v_r)$ = d$(v_r, v_s)$ = 2 and 1≤ i <j<r<s ≤3n. Let T= N$[v_i]$ ∪ N$[v_j]$, |T| = S. Then the subgraph induced by the vertices of V – T is$P_m \cup P_k$, where m = 3l, l ≥1, k = 3n – 8 – 3l or m = 3l – 1, l ≥1, k = 3n – 3l – 7 or m = 3l – 2, l ≥1, k = 3n – 6 – 3l. Let S be any strong efficient dominating set of H, $v_j, v_r \in$ S.

**Subsubsubcase (3d5a):** Let m = 3l, k = 3n – 8 – 3l = 3n – 3l – 9+1= 3(n – l – 3)+1. Then $\gamma_{se}(P_{3l})$= land $\gamma_{se}(P_{3n-3l-8})$= n – l – 2. Therefore |S| = 2+ l + n – l – 2= n =$\gamma_{se}(G)$. Hence $bc_{se}(G) \geq 3$.

**Subsubsubcase (3d5b):** Let m = 3l – 1, k = 3n – 3l – 8+1 = 3n – 3l – 9+2 = 3(n – l – 3)+2. Then $\gamma_{se}(P_m)$=$\gamma_{se}(P_{3l-1})$ = $\gamma_{se}(P_{3(l-3)+2})$= l – 1 and $\gamma_{se}(P_k)$ = n – l – 1. Hence |S| = 2+l – 1+n – l – 1= n =$\gamma_{se}(G)$. Hence $bc_{se}(G) \geq 3$.

**Subsubsubcase (3d5c):** Let m = 3l − 2, k = 3n – 8 – 3l+2 = 3n – 3l – 6= 3(n –l – 2). Then $\gamma_{se}(P_m) = \gamma_{se}(P_{3(l-1)+1})$ = l and $\gamma_{se}(P_k)$=$\gamma_{se}(P_{3(n-l-2)})$ = n – l – 2. Therefore $\gamma_{se}(H)$ = 2+ l +n – l – 2 = n =$\gamma_{se}(G)$. Hence $bc_{se}(G) \geq 3$.

**Subsubsubcase (3d5d):** Suppose d$(v_j, v_r)$ ≥ 3 or $v_j, v_r$ are adjacent. Then the proof is similar to above three sub-sub-sub-cases and hence $bc_{se}(G) \geq 3$.

**Subsubsubcase (3d5e):** Suppose i <r<s<j or i<r<j<s. Then $v_i$ *or* $v_j$ and $v_r$ *or* $v_s$ must belong to S. They strongly efficiently dominate 8 vertices. The subgraph induced by remaining 3n – 8 vertices is the union of paths. Here n – 2vertices are needed to strongly efficient dominate 3n – 8 vertices. Hence $\gamma_{se}(H)$ = n – 2+2 = n =$\gamma_{se}(G)$. Further it is verified that no set with less than n elements is a $\gamma_{se}$- set of H. From all the above cases, $bc_{se}(G) \geq 3$. Therefore $bc_{se}(G) = 3$.

**Theorem 2.4:** $bc_{se}(C_{3n}) = 3$, for all n ≥1.

**Proof:** Let G =$C_{3n}$, for all n ≥1.

**Case (1):** Suppose one edge is added with $C_{3n}$. Let H = G + e. Suppose S is a strong efficient dominating set of H.

**Sub case (1a):** Let e = $v_i v_j$ and d($v_i, v_j$) = 3k, 1≤k≤n – 1. Then $v_j$=$v_{3k+1}$ and hence deg$v_i$ = deg $v_j$ = 3 = Δ(G). Hence either $v_i$ *or* $v_j$∈ S. Let $v_i$∈ S. Then $v_{i+3}$,$v_{i+6}, \dots, v_{3k-2}$ belong to S. Hence$v_{3k}$∉ S. If $v_{3k}$∈S, then $|N_s[v_{3k-1}] \cap S|$ = $|\{v_{3k-2}, v_{3k}\}|$=2> 1, a contradiction. This subcase does not exist. The proof is similar if $v_j$∈ S.

**Sub case (1b):** Suppose d($v_i, v_j$) = 3k – 1, 1 ≤k≤n – 1. Then $v_j$=$v_{3k}$, $S_1$= {$v_i$,$v_{i+3}$, $v_{i+6}$, ... , $v_{i+3k-3}$, $v_{i+3k}$, $v_{i+3k+3}, \dots, v_{i+3n-3}$} and $S_2$ = {$v_j$,$v_{j+3}$, $v_{j+6}, \dots, v_{j+3k-3}$, $v_{j+3k}$, $v_{j+3k+3}$, ... , $v_{j+3n-3}$} are the only strong efficient dominating set of G. Therefore | $S_1$ |=|$S_2$ |=n and hence $\gamma_{se}(H)$= $\gamma_{se}(G)$. Hence $bc_{se}(G) \geq 2$.

**Sub case (1c):** Suppose d($v_i, v_j$) = 3k+1, k≥1. Then $v_j$=$v_{3k+2}$ and $S_1$,$S_2$ (as in the above subcase(1b)) are the only strong efficient dominating sets in this sub-case also. Hence $bc_{se}(G)$≥ 2.

**Case (2):** Suppose two edges $e_1$ and $e_2$ are added with G. Let H = G+{$e_1$ , $e_2$}. Let S be a strong efficient dominating set of H.

**Sub case (2a):** Let $e_1 = v_i v_j$ and $e_2 = v_r v_s$.Then $v_i, v_j, v_r$ *and* $v_s$ are the only maximum degree vertices. Therefore $v_i$ *or* $v_j$ and $v_r$ *or* $v_s$ belongs to S.They strongly efficient dominate 8 vertices of H.Then the sub-graph induced by the remaining 3n – 8 vertices is the union of paths. Here n – 2 vertices are needed to strongly efficient dominate them. Further it is verified that no set with n elements is a strong efficient dominating set of H. From the above all cases, $bc_{se}(G) \geq 3$.

**Case (3):** Suppose three edges are added with $C_{3n}$. Let H = G + {$e_1$ , $e_2$ , $e_3$}. Let $e_1 = v_i v_{i+2}$ , $e_2 = v_i v_{i+3}$, $e_3 = v_i v_{i+4}$. Then S= {$v_i, v_{i+6}, v_{i+9}, \dots, v_{i+3n-3}$}is the unique strong efficient dominating set of H. Therefore |S|=1+$\frac{i+3n-3-i-6}{3}$ +1 = n – 1 < $\gamma_{se}(G)$. Hence $bc_{se}(G)$ = 3.

**Theorem 2.5:** Let G ⊙ $K_1$= $G'$ be a connected strong efficient graph. Then $bc_{se}(G')$ = 1.

**Proof:** Let $G'$ = G⊙ $K_1$. Let V($G'$) = {$v_i, u_i$: 1 ≤ $i$ ≤ n} and E($G'$) = E(G) ∪{$v_i u_i$: 1 ≤ $i$ ≤ n}. Let S be a $\gamma_{se}$- set of $G'$. Then deg $v_i \geq 2$ and deg $u_i = 1$, 1 ≤ $i$ ≤ n. If none of $v_i$ belongs to S, then all $u_i$ belong to S. But $u_i$ cannot strongly dominate corresponding $v_i$. Hence no element in S to strongly efficiently dominates $v_i$, a contradiction. Therefore, at least one $v_j$ for some j, 1 ≤ $j$ ≤ n belongs to S. If all $v_i$ belong to S, then one adjacent vertex $v_k$ whose degree is less than or equal to that of $v_i$ is strongly dominated by two vertices $v_i$ and $v_k$, a contradiction. Therefore there exist some vertices do not belong to S. Let $v_s$ be such vertex. Clearly $v_s$∉ S. Therefore $u_s$∈ S. Let $v_i$ be one of the maximum degree vertex and $v_i$∈ S. Let e = $v_i u_s$ and H = $G'$+e. Then $S'$= S – {$u_s$} and |$S'$| = |S| – 1. Therefore $\gamma_{se}(H)$=$\gamma_{se}(G')$– 1 and $bc_{se}(G')$ = 1.

**Illustration 2.6:**

Consider the following graph G

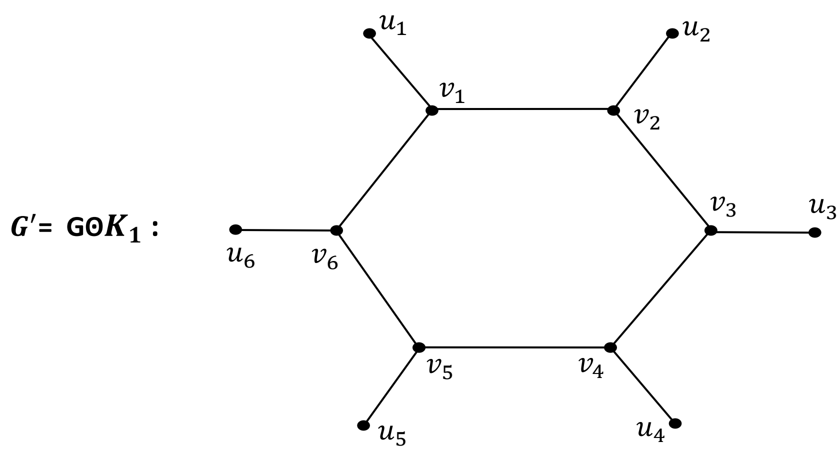


**Figure 2.2.**

Let $G'$ = G ʘ $K_1$. Let G = $C_6$. Then $\{v_1,v_4,u_2,u_3,u_5,u_6\}$ is a strong efficient dominating set of $G'$. Therefore $\gamma_{se}(G')$ = 6. Let e = $v_1u_6$. Let H = $G'$ + e. Then $\{v_1,v_4,u_2,u_3,u_5\}$ is the unique strong efficient dominating set of H. Therefore $\gamma_{se}$(H) = 5. Hence $bc_{se}(G')$ = 1.

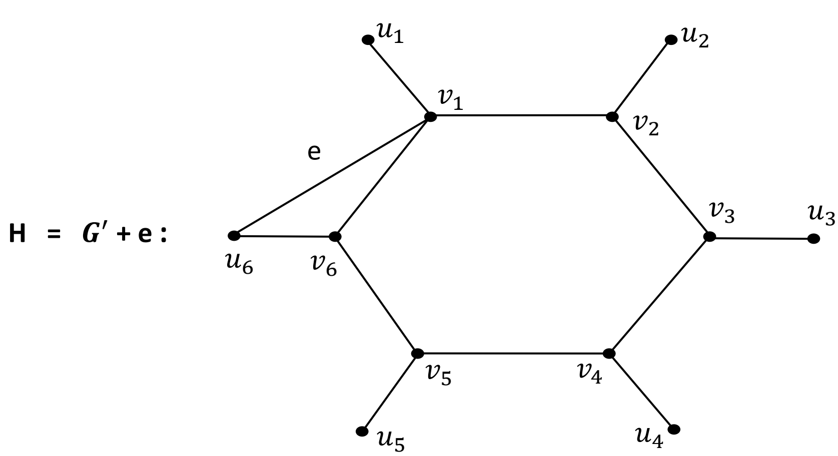


**Figure 2.2.**

**Theorem 2.7:** $bc_{se}(K_{n,n} - 1F)$= n, n ≥2.

**Proof:** Let G = $K_{n,n} - 1F$. Let V(G) = $\{v_i, u_i : i = 1,2, ...,$n} and E(G) = $\{v_i u_j$: $1 \leq i, j \leq$ n and i ≠j}. Then $\{v_i u_i : 1 \leq i \leq$ n} are the strong efficient dominating set of G and $\gamma_{se}(G)$ = 2. To reduce the strong efficient domination number a full degree vertex is needed, since$\gamma_{se}(G)$ =1 if and only if G contains a full degree vertex. Since degree of each vertex is n – 1, n new edges should be added to have a full degree vertex. Let $v_i$∈V(G). Then S = $\{v_i u_i, v_i v_l$: $1 \leq l \leq$ n and i ≠ l}. Let H be a new graph obtained by adding these n edges of S. Then deg$v_i$ = n+n – 1 = 2n – 1. Therefore $v_1$ is the full degree vertex and $\gamma_{se}$(H) =1. Hence $bc_{se}(G)$ = n, n ≥ 2.

**Theorem 2.8:**

$bc_{se}$(T(n,m)) =$\begin{cases} 1 \text{ } if \text{ } n = 3k - 1, k \geq 1 \\ 3 \text{ } if \text{ } n = 3k + 1, k \geq 1 \end{cases}$, where n ≠ 3k, n ≥ 1, and k≥ 1.

**Proof:** Let G = T(n,m). Let V(G) = {v, $u_i, v_j : 1 \leq i \leq m$, 1≤ $j \leq n$} and E(G) = {v$u_i$,v$v_1$, $v_t v_{t+1} : 1 \leq t \leq n - 1$}.

**Case1:** Let n = 3k – 1, k ≥1. Then {v,$v_3, v_6, ..., v_{3k-3}, v_{3k-1}$} is the unique strong efficient dominating set of G. Therefore $\gamma_{se}$(H) = k+1. Let e = $vv_{3k-1}$. Let H = G + e. Then {v,$v_3, v_6, ..., v_{3k-3}$} is the unique strong efficient dominating set of H. Therefore $\gamma_{se}$(H) = k. So, $\gamma_{se}$(H) <$\gamma_{se}$(G). Hence $bc_{se}$(G) = 1.

**Case2:** Let n = 3k+1, k≥ 1. Then {v,$v_3, v_6, ..., v_{3k-3}, v_{3k}$} is the unique strong efficient dominating set of G. Therefore $\gamma_{se}$(H) = k+1. Let $e_1$ = $vv_{3k+1}$, $e_2$ = $vv_{3k}$ , $e_3$ = $vv_{3k-1}$. Let H = G + {$e_1, e_2, e_3$}. Then {v,$v_3, v_6, ..., v_{3k-3}$} is the unique strong efficient dominating set of H. Therefore $\gamma_{se}$(H) = k. So, $\gamma_{se}$(H) <$\gamma_{se}$(G). Hence $bc_{se}$(G) ≤ 3. ----- (1)
As, in the proof of theorem 2.3, three edges are needed to reduce the strong efficient domination number of G. Hence $bc_{se}$(G) ≥ 3.----- (2). From (1) and (2) we get, $bc_{se}$(G) = 3.

**Illustration 2.9:** Consider the following graph G = T(6,5)
Let G = T(6,5). Then {v, $v_3$,$v_5$} is the unique strong efficient dominating set of G. Therefore, $\gamma_{se}$(G) = 3. Let e = v$v_5$. Let H = G + e be given in the following Figure 2.3. Then {v,$v_3$} is the unique strong efficient dominating set of H. Therefore, $\gamma_{se}$(H) = 2< $\gamma_{se}$(G). Hence $bc_{se}$(G) = 1.

**Theorem 2.10:** Let $G_n$ be a twig graph. Then $bc_{se}(G_n)$ = 1, for every n ≥4.
**Proof:** Let G = $G_n$ be a twig graph. Let V(G) = $\{v_i, w_j, u_j$: $1 \leq i \leq n, 2 \leq j \leq n - 1\}$ and E(G) = $\{v_j u_j, v_i v_{i+1}, v_j w_j$: $1 \leq i \leq n - 1, 2 \leq j \leq n - 1\}$.

**Case1:** Let G = $G_n$, where n=3k, k ≥2. Then S = $\{v_2, v_5, ..., v_{3k-1}, u_4, u_7, ..., u_{3k-2}$, $w_4, w_7, ..., w_{3k-2}$ , $u_3, u_6, ..., u_{3k-3}$, $w_3, w_6, ..., w_{3k-3}\}$ is the unique strong efficient dominating set of G. Therefore $\gamma_{se}$(G) = |S| = 5k –4, for all k≥2. Let e = $v_{n-1}u_{n-2}$. Let H = G + e. Then T = $\{v_2, v_5, ..., v_{3k-1}, u_4, u_7, ..., u_{3k-2}$, $w_4, w_7, ..., w_{3k-2}$ , $u_3, u_6, ...$ , $u_{3k-3}$, $w_3, w_6$,..., $w_{3k-3}\}$ is the unique strong efficient dominating set of H. Therefore $\gamma_{se}$(H) = |T| = |S| –1= 5k – 5, ∀ k≥2.So, $\gamma_{se}$(H) <$\gamma_{se}$(G). Hence $bc_{se}$(G) = 1, ∀n=3k, k≥2.

**Case 2:** Let G = $G_n$, where n=3k – 1, k≥2. Then S =$\{v_1, v_3, ..., v_{3k-3}, v_{3k-1}, u_2, u_5, ...$,

$u_{3k-4}, w_2, w_5, \dots, w_{3k-4}, u_4, u_7, \dots, u_{3k-2}, w_4, w_7, \dots, w_{3k-2}\}$ is the unique strong efficient dominating set of G. Therefore $\gamma_{se}$(G) = |S| = 5k – 3, ∀ k≥ 2. Let e=$v_n u_{n-1}$. Let H = G + e. Then T = $\{v_1, v_3, \dots, v_{3k-3}, u_2, u_5, \dots, u_{3k-4}, w_2, w_5, \dots, w_{3k-4}, u_4, u_7, \dots, u_{3k-2}, w_4, w_7,$ ..., $w_{3k-2}\}$ is the unique strong efficient dominating set of H. Therefore $\gamma_{se}$(H) = |T| = |S| – 1= 5k – 4, ∀ k≥2. So, $\gamma_{se}$(H) <$\gamma_{se}$(G). Hence $bc_{se}$(G) = 1, ∀n=3k – 1, k ≥ 2.

**Case 3:** Let G = $G_n$, where n=3k+1, k≥ 2. Then $S_1$ = $\{v_1, v_3, \dots, v_{3k}, u_2, u_5, \dots, u_{3k-1}, w_2, w_5, \dots, w_{3k-1}, u_4, u_7, \dots, u_{3k-2}, w_4, w_7, \dots, w_{3k-2}\}$ and $S_2$ = $\{v_2, v_5, \dots, v_{3k+1}, u_4, u_7, \dots, u_{3k-2}, w_4, w_7, \dots, w_{3k-2}, u_3, u_6, \dots, u_{3k}, w_3, w_6, \dots, w_{3k}\}$ are strong efficient dominating sets of G. Hence$\gamma_{se}$(G) = |$S_1$| = |$S_2$ | = 5k – 1, ∀ k≥ 1. Let e =$v_n u_{n-1}$. Let H =G + e. Then T = $\{v_2, v_5, \dots, v_{3k-1}, u_3, u_6, \dots, u_{3k}, w_3, w_6, \dots, w_{3k}, u_4, u_7, \dots, u_{3k-2}, w_4, w_7, \dots, w_{3k-2}\}$ is the unique strong efficient dominating set of H. Therefore $\gamma_{se}$(H) = |T| = |$S_1$| – 1 = |$S_2$ | – 1= 5k – 2, ∀ k≥1.So,$\gamma_{se}$(H)<$\gamma_{se}$(G). Hence $bc_{se}$(G) = 1, ∀n=3k+1, k ≥ 1. From the above all cases, $bc_{se}$(G) = 1.

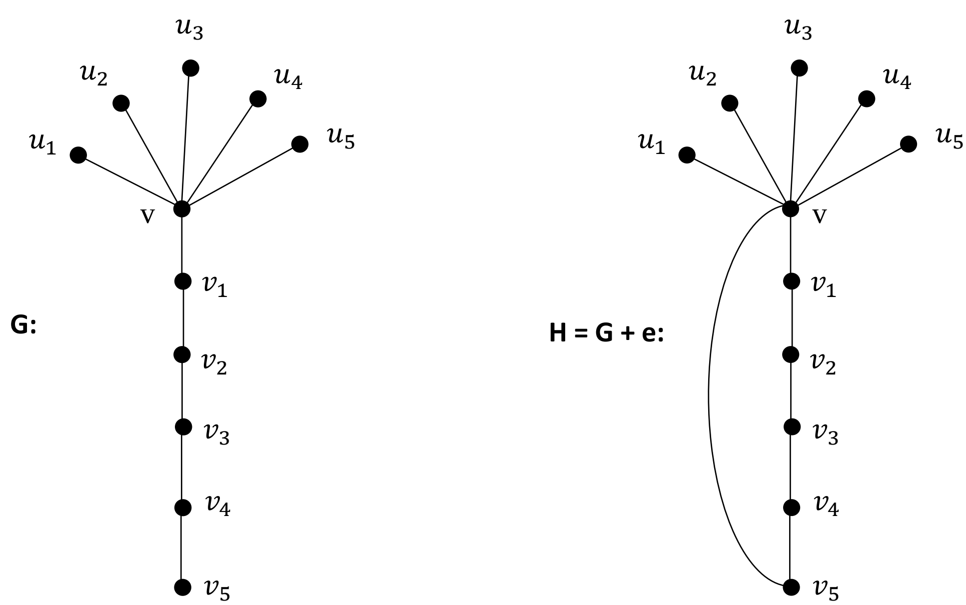


**Figure 2.3.**

**Theorem 2.11:** $bc_{se}(D_{r,s})$ = 1, where, s ≥ 1.

**Proof:** Let G = $D_{r,s}$, where s ≥ 1 and r ≥ s. Let V(G) = {u,v,$u_i, v_j$: $1 \le i \le r$ and $1 \le j \le s\}$ and E(G) = {uv, u$u_i$, $vv_j$: $1 \le i \le r$ and $1 \le j \le s\}$. Then $\gamma_{se}$(G) = s+1. Let e = {u$v_j$: $1 \le j \le s\}$. Let H = G + e. Then T = {u, $v_k$: $1 \le k \le s$ and k ≠ j} is the unique strong efficient dominating set of H. Therefore, |T| = s. So, $\gamma_{se}$(H) <$\gamma_{se}$(G). Hence $bc_{se}$(G) = 1.

## 4. CONCLUSION

Thus in this paper the authors studied the strong efficient co-bondage number of some standard and special graphs. Similar studies can be made on this type for various derived graphs.

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