TECHNICAL TRANSACTIONS | **CZASOPISMO TECHNICZNE**
FUNDAMENTAL SCIENCES | NAUKI PODSTAWOWE

2-NP/2015

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ON THE BEGINNING OF TOPOLOGY IN LWÓW

POCZĄTKI TOPOLOGII WE LWOWIE

Abstract

We provide one of the first surveys of results in the area of topology by representatives of the Lvov School of mathematics and mathematicians related to the University of Lvov. Viewed together, these results show the importance of this school in the creation of topology.

Keywords: *topology*, *history of mathematics in Poland*, *Lvov School of Mathematics*

Streszczenie

W artykule dokonamy jednego z pierwszych przeglądów wyników Lwowskiej Szkoły Matematycznej z zakresu topologii w celu ukazania znaczenia tej szkoły w tworzeniu topologii.

Słowa kluczowe: *topologia*, *historia matematyki w Polsce*, *Lwowska Szkoła Matematyczna*

DOI: 10.4467/2353737XCT.15.211.4416

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By now, the history of the Lvov School of Mathematics has already been well studied by historians of mathematics. The monograph by R. Duda [18] recently translated into English is an exploit in this direction. On the other hand, there are many publications on the history of topology in Poland. This raises a natural question: can one find a description of the history of Lvov topology at the intersection of the above groups of publications? Is it enough, say, to provide topological study as a part of the history of Lvov mathematical school? We think that the answer to this question is negative. Therefore we focus on topological grounds of the Lvov mathematics (considered in the period from the late 19th century to the Second World War). We will draw attention to some points not previously observed and also add a certain consistency to the material. Clearly, mathematics, as do other sciences, has not only its history but also its geography. In the period of activity of the Lvov School of Mathematics the role of the latter was much higher than it is currently, when electronic means of communication greatly facilitate the cooperation of scientists at the distance and blur the links of scientific results to a specific territory.

Our presentation of the material is inevitably centered around personalities. Given the large amount of biographical literature, we focus mainly at mathematical results. Moreover, we are interested in the most important scientific contribution made in topology during the Lvov period of activity of the individuals discussed.

Józef Puzyna

Born in 1856 in Nowy Martynów, near Rohatyń. For over 30 years he taught mathematics at the University of Lwów. He served in various university administrative functions, was a rector of the university. He was the first who mentioned topology as a mathematical subject in Polish in his “Studya topologiczne” (“Topological studies”). The basic scientific direction for Puzyna was comprehensive analysis. The main achievement of Puzyna, namely, his monograph *Teorya funkcyi analitycznych* (Theory of analytic functions) contains the very first exposition in Polish of the foundations of set theory and set-theoretic topology. The contents of this book are described in more detail in [11–13].

Much of Puzyna’s book is concerned with topology of surfaces. Despite the fact that in the introductory chapters of the monograph he actively uses the language of set theory, the author prefers the traditional descriptive approach of topology of surfaces while presenting the theory of surfaces.

For the theory of surfaces in Puzyna’s monograph reader can refer to the article [12] in this volume.

Zygmunt Janiszewski

Born in Warsaw in 1888. He studied mathematics at the University of Zurich, Göttingen, Paris, Munich, Marburg, Graz. He attended lectures of famous mathematicians including J. Hadamard, E. Borel, H. Minkowski, Landau, C. Runge, D. Hilbert, Picard, E. Zermelo, A. Toeplitz, F.Bernstein, E. Goursat. His doctoral thesis *Sur les continues irreductibles entre deux points* was defended at the Sorbonne in 1911. Among the committee members were H. Lebesgue, H. Poincaré and Emile Borel. His fundamental work *On cutting the plane by continua* (O rozcinaniu płaszczyzny przez kontinua: [21]) is devoted to the topology of the plane. In particular, it contains the following statements on cutting of the plane:

1) The sum of two continua does not cut the plane provided that none of them cuts the plane and their intersection is either connected or empty.
2) The sum of two continua such that their intersection is disconnected cuts the plane.

The publication [21] in Polish is supplemented by an extended French abstract, which made the results more accessible to the Western readers. An interesting discussion on the influence of Janiszewski's results as well as results of other Polish topologists on the mathematical activity of some American mathematicians working approximately in the same period can be found in [2].

It was Janiszewski who suggested that only one direction of mathematical research should be chosen in Poland, namely, the set theory and (set-theoretic) topology.

Janiszewski died in 1920.

Wacław Sierpiński

Born in 1882 in Warsaw. Georgy Voronoy was one of his doctoral advisors. In 1908, W. Sierpiński was approved as a Privatdozent in mathematics at the University of Lvov.

One of his books written during the Lvov period, *Outline of the set theory* (in Polish), is closely connected to set-theoretic topology. His Lvov period is taken to last until the end of March, 1919; although Sierpiński was interned in the years 1914–1917 (in the list of lectures delivered at the University of Lvov in 1916/17 one can see the remark that "Prof. Sierpiński does not have lectures in this semester") and took the leave on demand in the Fall semester, 1918/19.

Therefore, the known important objects of fractal geometry such as Sierpiński gasket and Sierpiński carpet were introduced by a mathematician affiliated to the University of Lvov.

Among Sierpiński's topological results of this period one should mention also the universality of Sierpiński's carpet in the sense that it contains a topological image of any planar curve. A three-dimensional generalization of Sierpiński's carpet, namely, Menger curve, is a universal object for all three-dimensional curves. Further generalizations of Sierpiński's carpet are the m-dimensional sets μ_m^n in the n-dimensional Euclidean space. The sets μ_m^{2n+1} are universal spaces for n-dimensional sets; they are also model spaces for the so-called Menger manifolds [3].

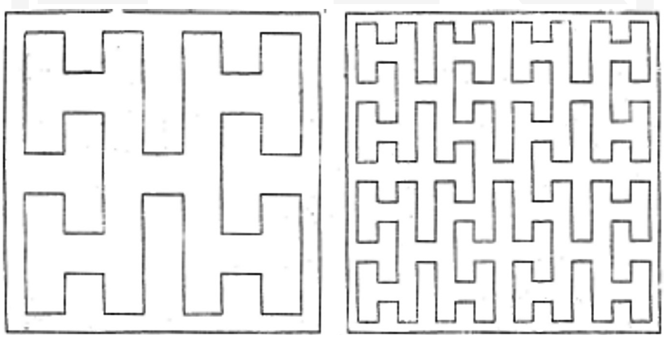

Fig. 1. Approximations of the universal curve. A picture from Sierpiński's paper

The modern theory of Menger manifolds was developed by many authors (see, e.g. [39, 40]). It is an *n*-dimensional counterpart of the theory of Hilbert cube manifolds, which itself provides an infinite-dimensional generalization of finite-dimensional manifolds.

One more result from this period concerns the space-filling curves, i.e. curves whose images contain squares. The object constructed in [48] is now called the Sierpiński curve. Note that Sierpiński continued investigations of Peano and Hilbert. The Sierpiński curve is an example of fractal curve; its approximations provide solutions of the Travelling Salesman Problem. Interesting information concerning space-filling curves can be found in [7].

W. Sierpiński died in 1969.

Józef Schreier

Born in Drohobycz in 1909. Schreier prepared his PhD thesis *On finite basis in topological groups* in 1932–1934. He was awarded his PhD in 1934, and Stefan Banach participated in the ceremony of award of degree as the advisor.

S. Ulam was the coauthor of eight publications with Schreier. They worked both in the topological group and topological semigroup theory. Stanisław Ulam recognized the importance of investigation of topological-algebraic objects.

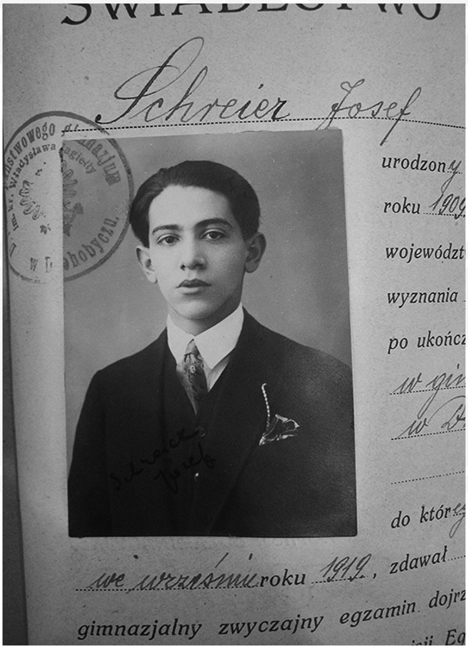

Fig. 2. Józef Schreier (Lviv Discrict Archiv)

In [46] Schreier and Ulam proved that every automorphism of the group S_ω of permutations with finite supports of a countable set is an inner automorphism, i.e. there exists an element *s* in *S* such that $x = sxs^{-1}$, for any *x* in *S*. Here, S_ω is regarded as a topological group with respect to the topology of pointwise convergence.

Some of publications by Schreier and Ulam concern the notion of base of a topological (semi)group. A detailed exposition of this direction of their investigation can be found in [19].

J. Schreier died in 1943.

Stefan Mazurkiewicz

Born in Warsaw, in 1888. Mazurkiewicz's doctorate (University of Lvov, 1913) was supervised by Wacław Sierpiński. The results concerned the square-filling curves. One of his results consist in a complete proof (for $n = 2$) of statements announced by H. Lebesgue [35] (the proof given therein turned out to be incomplete):

1) every planar curve which fills a 2-dimensional domain necessarily contains points of multiplicity 3;
2) there is a curve that fills a 2-dimensional domain and such that the multiplicity of its points is at most 3.

Also, it was proved in this paper that every planar nowhere dense continuum can be represented as the image of Cantor set such that the preimage of every point consists of at most 2 points.

Mazurkiewicz left Lvov for Warsaw in 1919 but still kept in contact with mathematicians from Lvov afterwards.

He died in Grodzisk Mazowiecki, in 1945.

Stefan Banach

Born in Kraków, in 1892. His Ph.D thesis *Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales* (University of Jan Kazimierz, Lvov, 1920) contained fundamental results in functional analysis.

He is recognized mostly because of his fundamental results in functional analysis. However, some of Banach's achievements are closely connected to set-theoretic topology and topological algebra. As examples, we mention the open mapping theorem, Banach--Alaoglu theorem, and Banach-Stone theorem. The fixed point theorem, although it belongs to metric geometry rather than topology, still finds great use even in solving purely topological problems.

We should also mention the method of proving the existence of categories of objects with specified properties. It became one of the hallmarks of the Lvov mathematical school.

S. Banach died in Lvov, in 1945.

Kazimierz Kuratowski

Born in Warsaw, in 1896. His Lvov period lasted for 6 years, until he left Lvov for Warsaw in 1934. The total number of works written by Kuratowski in the Lvov period is more than thirty, so we will focus only on some of them.

The famous Knaster-Kuratowski-Mazurkiewicz lemma [22] is a result which is known to be equivalent to the Brouwer fixed point theorem. However, its formulation turns out to be convenient for the use in mathematical economics, in particular, in the market equilibrium theory.

The paper [25] contains the following result: for every Peano continuum, the fixed point property implies unicoherence. As a consequence, one can derive some of Janiszewski's results mentioned above. A consequence is the unicoherence property of the plane established earlier by Mazurkiewicz.

In the paper [26] Kuratowski provided a topological characterization of the 2-dimensional sphere. To this aim he introduced the class of Janiszewski spaces, namely the spaces X satisfying the property that every continuum cutting X is unicoherent.

In his paper [19] published in Studia Mathematica, Kuratowski demonstrated a purely topological nature of the following result of S. Banach [0]: in a topological group, every subgroup satisfying the Baire property is simultaneously open and closed.

Kuratowski made important contributions to the dimension theory. In [27] he developed some ideas of Witold Hurewicz and proved a characterization theorem for dimension of perfect subsets in compact spaces in terms of mappings.

It is important also to emphasize that the first volume of Kuratowski's fundamental monograph *Topologie* [31] was published during the author's Lvov years.

Kuratowski died in Warsaw, in 1980.

Juliusz Schauder

Born in Lvov, in 1899. He was awarded his PhD in 1923 with the thesis *The theory of surface measure*.

One of the most important of Schauder's topological achievements is his fixed point theorem for the convex compact sets in linear topological spaces [44]. This is a generalization of the classical Brouwer fixed point theorem to infinite-dimensional case. However, soon after this publication it turned out that the proof worked smoothly only in the locally convex case (see also [50]). The general case was formulated as an open problem in the Scottish book. This gave a start to numerous publications in this direction.

The general case was considered by R. Cauty in 2001. The very first proof by Cauty of Schauder conjecture contained a gap. T. Dobrowolski [9] elaborated on Cauty's proof in order to make it more accessible. In this way he discovered an essential mistake.

Later, Cauty developed the theory of algebraic neighborhood retracts that allowed for obtaining fairly general results on fixed points (e.g., for the so-called locally equiconnected spaces).

Also, Schauder proved some results on domain invariance in infinite-dimensional linear spaces [43]. In collaboration with the French mathematician Jean Leray, Schauder developed the theory of degree of some nonlinear maps of Banach spaces [36]. The degree is a homotopy invariant of maps.

Juliusz Schauder died in Lvov, in 1943.

Miron Zarycki

Born in 1889, in Tarnopol region. Under the influence of Sierpiński, Zarycki's interests in mathematics turned to the set theory and real analysis.

Zarycki was awarded PhD (University of Lvov, 1930) for the thesis *Quelques notions fondamentales de l'Analysis Situs au point de vue de l'Algèbre de la Logique* published in Fund. Math. [53]. This paper is, in a sense, a continuation of Kuratowski's research on the closure operator in topological spaces. In particular, Zarycki formulated axioms for the operations of boundary of sets in topological spaces. It turned out that the system of axioms obtained is equivalent to that formulated by Kuratowski [23] and therefore the notion of the boundary can be equivalently used for defining topological spaces. This result by Zarycki was cited not only in topology but also, because of the importance of the notion of boundary in ontology, in some philosophical papers [52].

Note that Zarycki was a vice-Dean while Stefan Banach was the Dean of Department of Physics and Mathematics, and served as Dean after Banach's death. Zarycki died in Lvov, in 1961.

Stanisław Ulam

Born in Lvov, in 1909. Published his first paper in Fundamenta mathematicae when he was 20 (the paper was written two years before its publication). PhD in 1933 at the Lvov Polytechnical School.

One of the most important of Ulam's contributions to topology is the famous Borsuk--Ulam theorem [3]. It states that for any continuous map of an n-dimensional sphere into an n-dimensional euclidean space there are two antipodal points with the same image. The result was conjectured by Ulam and proved by Borsuk. (More information on Ulam's activity can be found in the article by L. Bazylevych, I. Guran and M. Zarichnyi in this volume).

Ulam died in Santa Fe (USA), in 1984.

The table below shows the equivalence of some of the mentioned results to their combinatorial and covering versions. It is notable that two results in this table belong to mathematicians from Lvov.

Algebraic topology	Combinatorics	Set covering
Brouwer fixed-point theorem	Sperner's lemma	KKM-lemma
Borsuk–Ulam theorem	Tucker's lemma	Lusternik–Schnirelmann theorem

Fig. 3. Equivalence of several topological, combinatorial, and covering results (taken from open sources)

Concluding remarks

Given the richness and heterogeneity of topological results obtained by the Lvov mathematical school, we cannot claim our brief survey to be complete. It demonstrates, however, that the achievements of mathematicians from the Lvov mathematical school in topology are in some aspects comparable with those in functional analysis.

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