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# POSSIBILITY OF EXISTENCE OF TORSIONAL VIBRATIONS OF BEAMS IN LOW FREQUENCY RANGE

# MOŻLIWOŚCI WYSTĄPIENIA DRGAŃ SKRĘTNYCH W BELKACH W ZAKRESIE NISKOCZĘSTOTLIWOŚCIOWYM

A b s t r a c t

This paper discusses the problem of the existence of torsional natural modes in the low frequency range for a beam with a rectangular cross-section with different ratios between width and height and different lengths. The rectangular type of cross-section is chosen because of practical applications to describe the vibrations of some turbine blades. The analyses were conducted by applying an analytical model, but were verified using the finite element method.

*Keywords*: *torsional vibrations*, *low frequency analysis*, *natural vibrations*

S t r e s z c z e n i e

W artykule rozważono zagadnienie możliwości wystąpienia form własnych drgań skrętnych w zakresie niskoczęstotliwościowym belek o prostokątnym kształcie przekroju. Rozważono wpływ stosunku wysokości do szerokości przekroju oraz długości belki. Prostokątny kształt przekroju został wybrany ze względu na zastosowania praktyczne analiz do opisu drgań łopatek turbin niektórych wirników. Rozważania opierają się na modelu analitycznym, ale były również weryfikowane poprzez analizy metodą elementów skończonych.

*Słowa kluczowe*: *drgania skrętne*, *analiza niskoczęstotliwościowa*, *drgania własne*

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## 1. Introduction

When analysing one-dimensional continuous type structural elements with a straight line in the general case, combined torsional-bending vibrations can arise. If the cross-section has two axes of symmetry, the shear centre and the gravity centre of the cross-section are the same point, and the torsional vibrations and the bending ones are separated. Therefore the torsional vibration of the beam can be analysed separately [4, 5, 9].

The motion of a realistic structure, independent of its shape, can be described as a deformable body. But to simplify the analysis, simpler models of the elements are commonly used. One of the groups to be considered is known as one-dimensional structures e.g. beams, shafts, rods. Due to the realistic cross-section shape, the description of motion of a beam can be described under some assumptions by applying a suitable theory. The general two groups of models, bearing in mind shape of the cross-section, are those for thin-walled cross-section and the monolytic ones [1, 3–5, 9]. If one characteristic dimension of the cross-section (known as the thickness) is small enough in comparison with the second one (known as the width) the whole structure is called thin-walled. A detailed rule states that for a thin-walled cross-section the thickness is more than eight times smaller than the highest dimension measured along the middle-line of the cross-section between its two end-points. Moreover, this length should be more than eight times smaller than the length of the beam [9]. It should be noted that the rectangle type of cross-section, for any ratios between width and height of cross-section, is to be understood as a monolytic type cross-section [9].

For some engineering applications, due to shape of their cross-sections and type of external loadings, the one-dimensional elements of beam type can be excited to torsional vibrations. For such cases, the torsional modes often lie in the low frequency for the structure [7]. The low frequency range is connected with the structure being analysed and is the range of frequency for which the modal density is not so high (not more than 10) and the value of the modal overlap factor is approximately unity.

It is commonly used in engineering vibrations to classify the frequency range of analysis to regions of low- and high-frequency. It should be pointed that the frequency upper limit of low-frequency analysis, and frequency the lower limit for high-frequency analysis depends on the dynamic characteristic of the structure analysed. In particular, it is a function of two parameters: modal density and modal overlap factor [2, 6].

This paper discusses the problem of the existence of torsional natural modes in the low frequency range for a beam with a rectangular cross-section with different ratios between width and height and different lengths. The low frequency range analysis is understood as taking into account a few of the lowest natural modes. The rectangular type of cross-section is chosen because of practical applications.

## 2. Natural vibrations of a beam

### 2.1. Torsional vibration

When the cross-section of a beam is a monolithic type with two axes of symmetry the equation of motion for natural torsional vibrations is independent of the equations of motion of bending vibrations, and has the form (1), where: $\varphi(x, t)$ – angle of torsion of the cross-section, $G$ – shear modulus,  – material density, $J_s$ – equivalent moment of inertia of cross-section due to torsion, $J_0$ – polar moment of inertia of the cross-section.

$$\rho J_0 \frac{\partial^2 \varphi(x,t)}{\partial t^2} - GJ_s \frac{\partial^2 \varphi(x,t)}{\partial x^2} = 0 \tag{1}$$

If the cross-section is circular in shape, the equivalent moment of inertia of cross-section due to torsion $J_s$ is equal to the polar moment of the inertia of cross-section $J_0$ and the equation of vibrations of the beam (1) is simplified to form (2).

$$\frac{\partial^2 \varphi(x,t)}{\partial t^2} - \frac{G}{\rho} \frac{\partial^2 \varphi(x,t)}{\partial x^2} = 0 \tag{2}$$

Differential equations for torsional vibrations (1) or (2) can be solved to find the values of natural frequencies and mode shapes after taking into account two boundary conditions. These can be formulated by giving suitable values of:

– angle of torsion $\varphi(x, t)$ for a given cross-section,
– value of torsional moment $M_s = GJ_s \frac{\partial \varphi}{\partial x}$ for a given cross-section.

For a non-circular cross-section during the process of torsion the deformation of cross-sections is observed (sometimes called warping). When analysing this process, the equivalent moment of inertia of cross-section due to torsion $J_s$ can be analytically formulated for the cross-section under consideration. If the rectangular cross-section is analysed, it can be determined approximately based on formula (3) [8–11], where: $b$ – width of cross-section, $h$ – height of cross-section. The formula can be simplified to form (4) [8, 10].

$$J_s = b^3 h \left( \frac{1}{3} - \frac{64}{\pi^5} \frac{b}{h} \sum_{m=1,3,5,\ldots}^{\infty} \frac{\tanh\left( \frac{m\pi}{2} \frac{h}{b} \right)}{m^5} \right) \tag{3}$$

$$J_s = bh^3 \frac{1}{3} \left( 1 - 0.63 \frac{h}{b} + 0.052 \left( \frac{h}{b} \right)^5 \right) \tag{4}$$

### 2.2. Bending vibrations

The equation of motion of a bending vibration of a beam based on the assumption of Bernoulli-Euler theory, for the case independent from the torsional one has the form (5), where: $w(x, t)$ – transversal displacement, $E$ – Young modulus, $J_y$ – moment of inertia of cross-section, $A$ – area of cross-section.

$$\rho A \frac{\partial^2 w(x,t)}{\partial t^2} + EJ_y \frac{\partial^4 w(x,t)}{\partial x^4} = 0 \tag{5}$$

The differential equation of the bending vibration (4) can be solved to find the values of natural frequencies and mode shapes after taking into account four boundary conditions. These can be formulated by giving the suitable values of:

– transversal displacement $w(x, t)$ for a given cross-section,
– slope angle $\varphi = -\frac{\partial w}{\partial x}$ for a given cross-section,
– value of bending moment $M_g = -EJ_y \frac{\partial^2 w}{\partial x^2}$ for a given cross-section,
– value of shear force $T = -EJ_y \frac{\partial^3 w}{\partial x^3}$ for a given cross-section.

## 3. Natural vibrations of a beam

### 3.1. General case

The easiest form of parametric presentation of the results of the analysis is a definition of the ratio $r_{(1)}^{(1)}$ between the lower value of natural frequency for torsional vibrations $\omega_t^{(1)}$ to the lower value of natural frequency for bending vibrations $\omega_g^{(1)}$. In general, the ratio $r_{(n)}^{(m)}$ is defined by (8), where $\lambda_t^{(m)}$ and $\lambda_g^{(n)}$ are parameters whose values depend on the type of boundary conditions of the beam under consideration, $m$ is the mode number of torsional vibrations and $n$ is the mode number of bending ones [5, 12]. The values for the natural frequencies of a beam can be determined based on formulas: (6) – for torsion and (7) for bending [5, 12].

$$\omega_t^{(m)} = \lambda_t^{(m)} \sqrt{\frac{GJ_s}{\rho J_0}} \tag{6}$$

$$\omega_g^{(n)} = \left(\lambda_g^{(n)}\right)^2 \sqrt{\frac{EJ_y}{\rho A}} \tag{7}$$

$$r_{(n)}^{(m)} = \frac{\omega_t^{(m)}}{\omega_g^{(n)}} = \frac{\lambda_t^{(m)} l}{\left(\lambda_g^{(n)} l\right)^2} l \sqrt{\frac{G}{E}} \sqrt{\frac{J_s A}{J_0 J_y}} = \frac{\lambda_t^{(m)} l}{\left(\lambda_g^{(n)} l\right)^2} l \sqrt{\frac{1}{2(1+\nu)}} \sqrt{\frac{J_s A}{J_0 J_y}} \tag{8}$$

### 3.2. Clamped-free beam with rectangular and circular cross-section

Bearing in mind practical applications in turbine blades, the case of a clamped-free beam with a rectangular cross-section is analysed in detail as an example. A beam with length $l$ has the rectangular cross-section with width $b$ and height $h$. Moreover, the circular cross-section with diameter $D$ is considered for comparison of results. The boundary conditions for torsion have form (9) and for bending form (10).

$$\varphi|_{x=0}=0, \quad \frac{\partial \varphi}{\partial x}|_{x=l}=0 \tag{9}$$

$$w|_{x=0}=0, \quad \frac{\partial w}{\partial x}|_{x=0}=0, \quad \frac{\partial^2 w}{\partial x^2}|_{x=l}=0, \quad \frac{\partial^3 w}{\partial x^3}|_{x=l}=0 \tag{10}$$

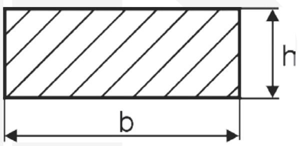
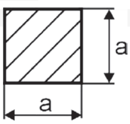
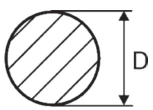
For this case, the values of $\lambda_t^{(1)}$ and $\lambda_g^{(1)}$ can be calculated from relationship (12). Values for the other cases can be found e.g. in [12]. Based on these formulas and relationship (4) for the three sections – rectangle, square and circle – the formula for ratio $r_{(1)}^{(1)}$ can be written in the forms given in Table 1 for material described by a Poisson ratio equal to ν = 0.29 (e.g. steel, aluminium). For a rectangular cross-section, the non-dimensional parameter s (height/width) is useful for parametric description of the cross-section shape (11). Value $s = 1$ denotes a square cross-section.

$$s=\frac{h}{b} \tag{11}$$

$$\lambda_t^{(1)} l=1.5708, \quad \lambda_g^{(1)} l=1.8751, \quad \lambda_g^{(2)} l=4.6941, \quad \lambda_g^{(3)} l=7.854 \tag{12}$$

Table 1

**Formulas for ratio $r_{(1)}^{(1)}$ for chosen cross-section shapes**

| SHAPE | FORM | RATIO $r_{(1)}^{(1)}$ |
|---|---|---|
| RECTANGLE |  | $3.357\frac{l}{\sqrt{b^2+h^2}}\sqrt{\frac{1}{3}\left(1-0.63\frac{h}{b}+0.052\frac{h^5}{b^5}\right)}=$ $3.357\frac{1}{\sqrt{1+s^2}}\sqrt{\frac{1}{3}\left(1-0.63\,s+0.052s^5\right)}\,\frac{l}{b}$ |
| SQUARE |  | $0.891\frac{l}{a}$ |
| CIRCLE |  | $1.119\frac{l}{D}$ |

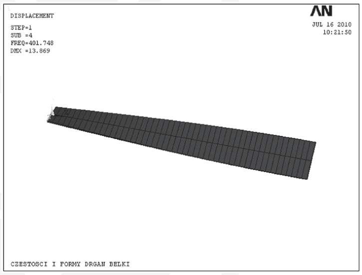
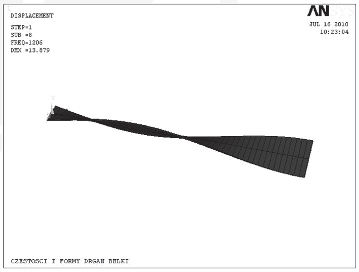
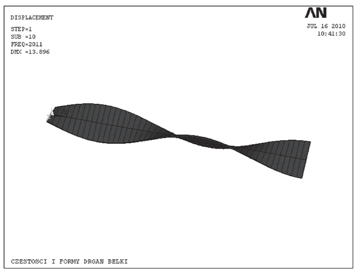
## 4. Analysis of existence of torsional modes in low frequency

### 4.1. FEM simulation

The simulations were performed for a free-fixed beam made of steel ($E$ = 2.1$E$ + 11 Pa, ν = 0.29, ρ = 7800 kg/m$^3$) with a length of 0.2 m and a rectangular cross-section (width 0.02 m, height 0.001 m). The length/width ratio is equal to ten, so the Bernoulli-Euler theory of bending can be applied. In the modal characteristic of the beam under consideration, there are two types of modes: bending (in two planes) and torsional. Only torsional modes are taken into account here.

Table 2

**Natural frequencies of torsional vibrations for the analysed beam**

<table>
<tr><th rowspan="2">MODE NO.</th><th colspan="2">FREQUENCY [Hz]</th><th rowspan="2">MODE SHAPE</th></tr>
<tr><th>THEORY</th><th>FEM</th></tr>
<tr><td>1</td><td>397</td><td>402</td><td>
</td></tr>
<tr><td>2</td><td>1191</td><td>1206</td><td>
</td></tr>
<tr><td>3</td><td>1985</td><td>2011</td><td>
</td></tr>
</table>

The first analysis was performed by applying the finite element package. The beam was modelled in the *Ansys* computer package using the *solid45* element.

The second results were obtained by the analytical formulas from the model discussed.

The values of the first three lowest natural frequencies of torsional type, obtained by the finite element method and analytical formulas, are given in Tab.1. These results are close to each other.

### 4.2. Influence of ratio height/width

The influence of the shape of the rectangular cross-section on the existence of torsional modes in the low frequency range is shown for the beam described above. The first three bending modes ($n$ = 1,2,3), and the first torsional mode ($m$ = 1) were taken into account. The independent parameter of the analysis was ratio $s$ (height/width) – (9). Hence three functions were analysed – $r_{(1)}^{(1)}(s)$, $r_{(2)}^{(1)}(s)$, $r_{(3)}^{(1)}(s)$. The functions are shown in Fig.1.

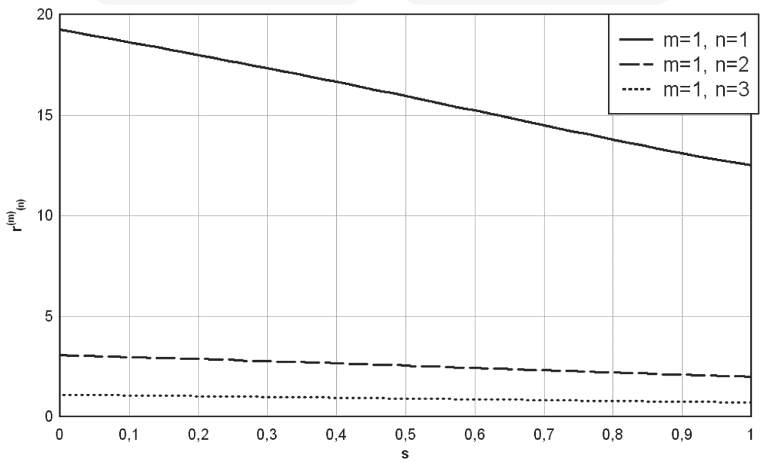



Fig. 1. Ratios $r_{(1)}^{(1)}(s)$ (top), $r_{(2)}^{(1)}(s)$, $r_{(3)}^{(1)}(s)$ (bottom) as functions of parameter s

It can be seen that the ratio for the first bending mode is much more than one for all values of parameter $s$. But for the third bending mode the ratio is approximately one. This means that the natural frequency for the third bending mode is almost the same as for the first torsional one for the beam with the geometry and material given.

### 4.3. Influence of length

The influence of the length of a rectangular cross-section on the existence of torsional modes in low frequency range is shown for the beam described above. The beam has a rectangular cross-section (width 0.02 m, height 0.001 m). The first three bending modes ($n$ = 1,2,3), and the first torsional mode ($m$ = 1) were taken into account. The independent parameter in this analysis was length $l$ (height/width). As above, three functions were analysed – $r_{(1)}^{(1)}(l)$, $r_{(2)}^{(1)}(l)$, $r_{(3)}^{(1)}(l)$. The functions are shown in Fig. 2.

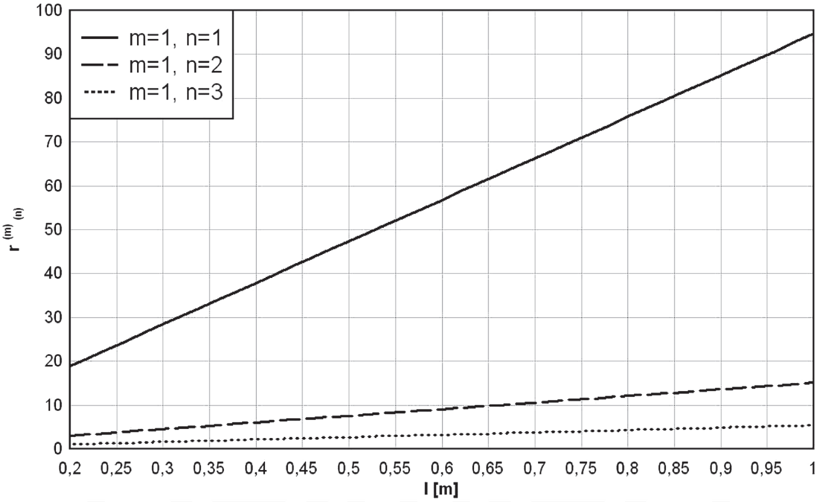



Fig. 2. Ratios $r_{(1)}^{(1)}(l)$ (top), $r_{(2)}^{(1)}(l)$, $r_{(3)}^{(1)}(l)$ (bottom) as functions of length l

It can be seen that the ratio for the first bending mode is much more than that for all values of parameter l. But for the third bending mode the ratio is approximately one for a short beam.

## 5. Conclusions

A fixed-free beam with a rectangular cross-section may exist in engineering structures, such as turbine blades. For some cross-section shapes, with an especially short length, the existence can be observed of torsional modes in the low frequency range for this beam.

For some cases of cross-sections, not analysed in the article, the theory of thin-walled beam must be applied in the analysis [1, 3–5, 9]. The rectangular cross-section type is not thin-walled for any values of the height/width ratio.

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