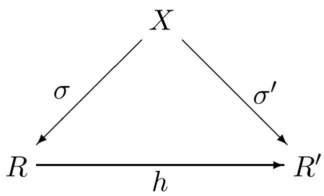
- $\sigma^*(x) = \sigma(x)$, for all $x \in X$, and
- $\sigma^*(T(t_0, s_0, s_1)) = I(\sigma^*(t_0), \sigma^*(s_0), \sigma^*(s_1))$, for all $t_0, s_0, s_1 \in \mathrm{Tm}(X)$.

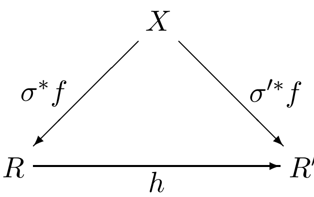
The morphisms $h : \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \to \langle\langle \mathbf{G}', \mathcal{S}'\rangle, \sigma'\rangle$ of $\mathrm{MOD}(X)$ are (strict coordinated) geometric morphisms $h : \langle \mathbf{G}, \mathcal{S}\rangle \to \langle \mathbf{G}', \mathcal{S}'\rangle$, that make the following diagram commute:



Given a mapping $f : X \to Y$ in **Sign** (i.e., $f : X \to \mathrm{Tm}(Y)$ in **Set**), the functor $\mathrm{MOD}(f) : \mathrm{MOD}(Y) \to \mathrm{MOD}(X)$ is defined, for all $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(Y)|$, by

$$\mathrm{MOD}(f)(\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle) = \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma^* f\rangle,$$

and $\mathrm{MOD}(f)(h) = h : \langle(\mathbf{G}, \mathcal{S}), \sigma^* f\rangle \to \langle(\mathbf{G}', \mathcal{S}'), \sigma'^* f\rangle$, for all $h : \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \to \langle\langle \mathbf{G}', \mathcal{S}'\rangle, \sigma'\rangle$ in $\mathrm{MOD}(Y)$. It is clear that this definition is sound, since the commutativity of the triangle displayed above implies the commutativity of



Finally, define, for all $X \in |\mathbf{Set}|$, the relation $\models_X \subseteq |\mathrm{MOD}(X)| \times \mathrm{SEN}(X)$, by setting, for all $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, with $\mathbf{G} = \langle P, L, I\rangle$, and all $(t_0, t_1)\ I\ (s_0, s_1) \in \mathrm{SEN}(X)$,

$$\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) \quad \text{iff} \quad \sigma^*(t_1) = \sigma^*(T(t_0, s_0, s_1)).$$

Let $\mathcal{AG} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$. $\mathcal{AG}$ is called the **institution of abstract geometry**. This terminology is justified by the following

Theorem 3. *The quadruple* $\mathcal{AG} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models\rangle$ *is an institution.*

Proof. Suppose $f : X \to Y$ be in **Sign**, $(t_0, t_1)\ I\ (s_0, s_1) \in \mathrm{SEN}(X)$ and $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(Y)|$, with $\mathbf{G} = \langle P, L, I\rangle$. Then

$$\begin{array}{ll} \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_Y \mathrm{SEN}(f)((t_0, t_1)\ I\ (s_0, s_1)) & \\ \quad \text{iff} & \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_Y (f^*(t_0), f^*(t_1))\ I\ (f^*(s_0), f^*(s_1)) \\ \quad \text{iff} & \sigma^*(f^*(t_1)) = \sigma^*(T(f^*(t_0), f^*(s_0), f^*(s_1))) \\ \quad \text{iff} & \sigma^*(f^*(t_1)) = I(\sigma^*(f^*(t_0)), \sigma^*(f^*(s_0)), \sigma^*(f^*(s_1))) \\ \quad \text{iff} & (\sigma^* f)^*(t_1) = I((\sigma^* f)^*(t_0), (\sigma^* f)^*(s_0), (\sigma^* f)^*(s_1)) \\ \quad \text{iff} & \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma^* f\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) \\ \quad \text{iff} & \mathrm{MOD}(f)(\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle) \models_X (t_0, t_1)\ I\ (s_0, s_1). \end{array}$$

Thus, the satisfaction condition holds and $\mathcal{AG}$ is an institution. □

5. The Ternary Ring $\mathbf{R} = \langle R, T, 0, 1\rangle$

In this section, we review the abstract properties of the ternary ring **R** that is formed by the coordinatization of an abstract geometry **G**, as presented in Section IV.6 of [3]. Moreover, we describe the reverse process of coordinatization, presented in Section IV.7 of [3], by which an abstract geometry **G** is associated with a ternary ring. The abstract properties of the coordinatizing ternary rings as well as this construction of an abstract geometry using the elements and the algebraic properties of a ternary ring will prove useful when we study the process of algebraization of abstract geometry in the final sections of this paper.

Let $\mathbf{G} = \langle P, L, I\rangle$ be an abstract geometry and let $\mathcal{S} = \langle (O, I, X, Y), R, \rho\rangle$ be a coordinate system for **G**. Let $T : R^3 \to R$ be the ternary operation on R defined by setting $T(x, m, b) = I(x, m, b)$, for all $x, m, b \in R$. Then, as is shown in Section IV.6 of [3], the operation T on R satisfies the following properties:

1. $T(0, m, b) = T(x, 0, b) = b$, for all $x, b, m \in R$.

2. $T(x, 1, 0) = T(1, x, 0) = x$, for all $x \in R$.

3. The equation $T(x, m, b) = T(x, m', b')$ has a unique solution in R, for all $m, m', b, b' \in R$, with $m \neq m'$.

4. The system of equations

$$\left\{ \begin{array}{rcl} T(a,x,y) & = & b \\ T(a',x,y) & = & b' \end{array} \right\}$$

has a unique solution in R, for all $a.a', b, b' \in R$, with $a \neq a'$.

5. The equation $T(a,m,x) = c$ has a unique solution in R, for all $a, m, c \in R$.

The ternary ring associated with $\mathbf{G}$ and the coordinatization $\mathcal{S}$ of $\mathbf{G}$ will be denoted by $\mathsf{R}(\mathbf{G}, \mathcal{S})$ or $\mathsf{R}_{\mathcal{S}}(\mathbf{G})$. More generally, an algebraic structure $\mathbf{R} = \langle R, T, 0, 1\rangle$, where T is a ternary operation satisfying properties 1-5 above, will be called a **Hall ternary ring**.

Suppose, next, that a Hall ternary ring $\mathbf{R}$ is given. Then an abstract geometry $\mathbf{G} = \langle P, L, I\rangle$ may be constructed as follows:

- $P = \{(x,y) : x, y \in R\}$;
- $L = \{\{(a,y) : y \in R\} : a \in R\} \cup \{\{(x, T(x,m,b)) : x \in R\} : m, b \in R\}$;
- The incidence relation is simply the membership relation, i.e., for all $(x,y) \in P$ and all $l \in L$, we have $(x,y)\ I\ l$ iff $(x,y) \in l$.

The following may now be established:

Theorem 4. *Given a Hall ternary ring, the structure* $\mathbf{G}$ *is an abstract geometry. Moreover, the Hall ternary ring associated with* $\mathbf{G}$ *under the coordinatization* $\mathcal{S} = \langle((0,0),(1,1),(1,0),(0,1)), R, i_R\rangle$ *coincides with the Hall ternary ring* $\mathbf{R}$.

The coordinated abstract geometry that is obtained in this fashion out of the given Hall ternary ring $\mathbf{R}$ will be denoted by $\mathsf{G}(\mathbf{R})$. According to Theorem 4, we have that $\mathsf{R}(\mathsf{G}(\mathbf{R})) = \mathbf{R}$.

6. The Institution $\mathcal{GA}$ of Geometric Algebra

In this section, the algebraic institution $\mathcal{GA}$, that will serve as the algebraic semantics of the institution $\mathcal{AG}$ of abstract geometry, will be constructed. Its models will be essentially Hall ternary rings and its sentences will be equations of terms over the language of Hall ternary rings. The fact that Hall ternary rings serve as coordinate rings for affine plane geometries is used in subsequent sections to construct mutually inverse interpretations between these two institutions.

Let **Sign** be the category of signatures of the institution $\mathcal{AG}$, defined in Section 4.

Define the functor EQ : **Sign** $\to$ **Set** as follows: Given a set X,

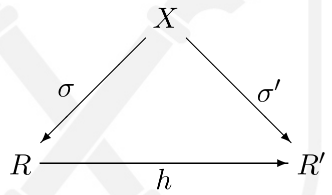
$$\mathrm{EQ}(X) = \{t_0 \approx t_1 : t_0, t_1 \in \mathrm{Tm}(X)\}$$

and, given $f : X \to \mathrm{Tm}(Y)$ in **Sign**, define

$$\mathrm{EQ}(f)(t_0 \approx t_1) = f^*(t_0) \approx f^*(t_1), \text{ for all } t_0, t_1 \in \mathrm{Tm}(X),$$

where $f^* : \mathrm{Tm}(X) \to \mathrm{Tm}(Y)$ is the unique extension of f on terms that was also defined in Section 4.

Furthermore, define the functor ALG : **Sign** $\to$ $\mathbf{Cat}^{\mathrm{op}}$ as follows: Given a set X, the category $\mathrm{ALG}(X)$ has as its objects all pairs $\langle \mathbf{R}, \sigma \rangle$, where $\mathbf{R}$ is a Hall ternary ring and $\sigma : X \to R$ an assignment of elements from the universe R of $\mathbf{R}$ to the variables in X. A morphism $h : \langle \mathbf{R}, \sigma \rangle \to \langle \mathbf{R}', \sigma' \rangle$ in $\mathrm{ALG}(X)$ is a Hall ternary ring homomorphism $h : \mathbf{R} \to \mathbf{R}'$, that makes the following diagram commute:



Moreover, given an $f : X \to \mathrm{Tm}(Y)$ in **Sign**, the corresponding functor $\mathrm{ALG}(f) : \mathrm{ALG}(Y) \to \mathrm{ALG}(X)$ sends an object $\langle \mathbf{R}, \sigma \rangle \in |\mathrm{ALG}(Y)|$ to the object $\mathrm{ALG}(f)(\langle \mathbf{R}, \sigma \rangle) = \langle \mathbf{R}, \sigma^* f \rangle$ and a morphism $h : \langle \mathbf{R}, \sigma \rangle \to \langle \mathbf{R}', \sigma' \rangle$ in $\mathrm{ALG}(Y)$ to $\mathrm{ALG}(f)(h) = h : \langle \mathbf{R}, \sigma^* f \rangle \to \langle \mathbf{R}', \sigma'^* f \rangle$ in $\mathrm{ALG}(X)$.

Finally, for every set X, define the **satisfaction relation**

$$\models_X \subseteq |\mathrm{ALG}(X)| \times \mathrm{EQ}(X)$$

by setting, for all $\langle \mathbf{R}, \sigma \rangle \in |\mathrm{ALG}(X)|$ and all $t_0, t_1 \in \mathrm{Tm}(X)$,

$$\langle \mathbf{R}, \sigma \rangle \models_X t_0 \approx t_1 \quad \text{iff} \quad \sigma^*(t_0) = \sigma^*(t_1).$$

The quadruple $\mathcal{GA} = \langle \mathbf{Sign}, \mathrm{EQ}, \mathrm{ALG}, \models \rangle$ is called the **institution of geometric algebra**, which is justified by the following:

Theorem 5. *The quadruple* $\mathcal{GA} = \langle \mathbf{Sign}, \mathrm{EQ}, \mathrm{ALG}, \models\rangle$ *is an institution.*

Proof. We check the satisfaction condition. Suppose that $f : X \to \mathrm{Tm}(Y)$ is a morphism in **Sign**, $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(Y)|$ and $t_0 \approx t_1 \in \mathrm{EQ}(X)$. Then, we have

$$\begin{array}{rll} \langle \mathbf{R}, \sigma\rangle \models_Y \mathrm{EQ}(f)(t_0 \approx t_1) & \text{iff} & \langle \mathbf{R}, \sigma\rangle \models_Y f^*(t_0) \approx f^*(t_1) \\ & \text{iff} & \sigma^*(f^*(t_0)) = \sigma^*(f^*(t_1)) \\ & \text{iff} & (\sigma^* f)^*(t_0) = (\sigma^* f)^*(t_1) \\ & \text{iff} & \langle \mathbf{R}, \sigma^* f\rangle \models_X t_0 \approx t_1 \\ & \text{iff} & \mathrm{ALG}(f)(\langle \mathbf{R}, \sigma\rangle) \models_X t_0 \approx t_1. \end{array}$$

□

7. $\mathcal{GA}$ is an Algebraic Semantics of $\mathcal{AG}$

In this section, it is shown that the institution $\mathcal{GA}$ of the equational logic of the Hall ternary rings, introduced in Section 6, constitutes an algebraic semantics of the institution $\mathcal{AG}$ of abstract geometry, introduced in Section 4. According to the theory of categorical abstract algebraic logic, an algebraic institution $\mathcal{A} = \langle \mathbf{Sign}', \mathrm{SEN}', \mathrm{MOD}', \models^{\mathcal{A}}\rangle$ is an algebraic institution semantics of an institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, \mathrm{MOD}, \models^{\mathcal{I}}\rangle$ if the corresponding π-institution $\pi(\mathcal{I}) = \langle \mathbf{Sign}, \mathrm{SEN}, C^{\mathcal{I}}\rangle$ is interpretable in $\pi(\mathcal{A}) = \langle \mathbf{Sign}', \mathrm{SEN}', C^{\mathcal{A}}\rangle$. Note that $C^{\mathcal{I}}$ is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$ by $\phi \in C^{\mathcal{I}}_\Sigma(\Phi)$ iff, for every model $M \in |\mathrm{MOD}(\Sigma)|$,

$$M \models^{\mathcal{I}}_\Sigma \Phi \quad \text{implies} \quad M \models^{\mathcal{I}}_\Sigma \phi.$$

A similar definition applies for $C^{\mathcal{A}}$, i.e., both $\pi(\mathcal{I})$ and $\pi(\mathcal{A})$ are the π-institutions whose closure systems are the closure systems induced by the semantical entailment systems of the corresponding institutions. The π-institution $\pi(\mathcal{I})$ is interpretable in $\pi(\mathcal{A})$ if there exists an interpretation $\langle F, \alpha\rangle : \mathcal{I} \to \mathcal{A}$, i.e., a functor $F : \mathbf{Sign} \to \mathbf{Sign}'$ and a natural transformation $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{SEN}' \circ F$, such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\phi \in C^{\mathcal{I}}_\Sigma(\phi) \quad \text{iff} \quad \alpha_\Sigma(\Phi) \subseteq C^{\mathcal{A}}_{F(\Sigma)}(\alpha_\Sigma(\phi)).$$

To show that $\mathcal{GA}$ is an algebraic semantics of $\mathcal{AG}$, we define the pair $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle : \mathcal{AG} \to \mathcal{GA}$ as follows: $\mathrm{I}_{\mathbf{Sign}} : \mathbf{Sign} \to \mathbf{Sign}$ is the identity functor on the common signature category of the two institutions. The natural transformation $\alpha : \mathrm{SEN} \to \mathcal{P}\mathrm{EQ}$ is defined by setting, for every set X and all $t_0, t_1, s_0, s_1 \in \mathrm{Tm}(X)$,

$$\alpha_\Sigma((t_0, t_1)\ I\ (s_0, s_1)) = \{t_1 \approx T(t_0, s_0, s_1)\}.$$

Lemmas 6 and 7, that follow, are supporting lemmas for showing the main equivalence establishing the interpretation property of $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle$. This equivalence is shown in Proposition 8. The main theorem, Theorem 9, simply restates the equivalence of Proposition 8 in the language of abstract algebraic logic.

Lemma 6. *Let X be a set, $\langle \mathbf{R}, \sigma \rangle \in |\mathrm{ALG}(X)|$ and $t_0, t_1, s_0, s_1 \in \mathrm{Tm}(X)$. Then*

$$\langle \mathbf{R}, \sigma \rangle \models_X t_1 \approx T(t_0, s_0, s_1) \quad \textit{iff} \quad \langle \mathsf{G}(\mathbf{R}), \sigma \rangle \models_X (t_0, t_1)\ I\ (s_0, s_1).$$

Proof.

$$\begin{array}{ll} \langle \mathbf{R}, \sigma \rangle \models_X t_1 \approx T(t_0, s_0, s_1) & \\ \quad \text{iff} & \sigma^*(t_1) = \sigma^*(T(t_0, s_0, s_1)) \\ \quad \text{iff} & \sigma^*(t_1) = I(\sigma^*(t_0), \sigma^*(s_0), \sigma^*(s_1)) \\ \quad \text{iff} & (\sigma^*(t_0), \sigma^*(t_0))\ I\ (\sigma^*(s_0), \sigma^*(s_1)) \\ \quad \text{iff} & \langle \mathsf{G}(\mathbf{R}), \sigma \rangle \models_X (t_0, t_1)\ I\ (s_0, s_1). \end{array}$$

□

Lemma 7. *Let X be a set, $\langle \langle \mathbf{G}, \mathcal{S} \rangle, \sigma \rangle \in |\mathrm{MOD}(X)|$ and $t_0, t_1, s_0, s_1 \in \mathrm{Tm}(X)$. Then*

$$\langle \langle \mathbf{G}, \mathcal{S} \rangle, \sigma \rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) \quad \textit{iff} \quad \langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma \rangle \models_X t_1 \approx T(t_0, s_0, s_1).$$

Proof.

$$\begin{array}{ll} \langle \langle \mathbf{G}, \mathcal{S} \rangle, \sigma \rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) & \\ \quad \text{iff} & \sigma^*(t_1) = \sigma^*(T(t_0, s_0, s_1)) \\ \quad \text{iff} & \langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma \rangle \models_X t_1 \approx T(t_0, s_0, s_1). \end{array}$$

□

Proposition 8. *Let X be a set. For all $\Phi \cup \{\phi\} \subseteq \mathrm{SEN}(X)$,*

$$\phi \in C_X^{\mathcal{AG}}(\Phi) \quad \textit{iff} \quad \alpha_X(\phi) \subseteq C_X^{\mathcal{GA}}(\alpha_X(\Phi)).$$

Proof. Assume, first, that $\phi \in C_X^{\mathcal{AG}}(\Phi)$. This means that, for all $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \Phi$ implies $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \phi$. Suppose, now, that $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$, and $\langle \mathbf{R}, \sigma\rangle \models_X \alpha_X(\Phi)$. Thus, $\langle \mathbf{R}, \sigma\rangle \models_X t_1 \approx T(t_0, s_0, s_1)$, for all $(t_0, t_1)\ I\ (s_0, s_1) \in \Phi$. Therefore, by Lemma 6, we have $\langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1)$, for all $(t_0, t_1)\ I\ (s_0, s_1) \in \Phi$. Hence, by hypothesis, we get that $\langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X \phi$. Again, using Lemma 6, we obtain that $\langle \mathbf{R}, \sigma\rangle \models_X \alpha_X(\phi)$. This proves that $\alpha_X(\phi) \subseteq C_X^{\mathcal{GA}}(\alpha_\Sigma(\Phi))$.

Assume, conversely, that $\alpha_X(\phi) \subseteq C_X^{\mathcal{GA}}(\alpha_X(\Phi))$. This means that, for all $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$, $\langle \mathbf{R}, \sigma\rangle \models_X \alpha_X(\Phi)$ implies $\langle \mathbf{R}, \sigma\rangle \models_X \alpha_X(\phi)$. Suppose, now, that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, such that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \Phi$. Thus, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1)$, for all $(t_0, t_1)\ I\ (s_0, s_1) \in \Phi$. Therefore, by Lemma 7, $\langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_1 \approx T(t_0, s_0, s_1)$, for all $(t_0, t_1)\ I\ (s_0, s_1) \in \Phi$. Hence, by hypothesis, we get that $\langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X \alpha_X(\phi)$. Again, using Lemma 7, we obtain that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \phi$. This proves that $\phi \subseteq C_X^{\mathcal{AG}}(\Phi)$.

□

Theorem 9. *The institution of Hall ternary rings $\mathcal{GA}$ is an algebraic semantics of the institution $\mathcal{AG}$ of abstract geometry.*

Proof. This is just a restatement of Proposition 8. □

8. Algebraization of $\mathcal{AG}$

In this section, it is shown that not only is $\mathcal{GA}$ an algebraic semantics of $\mathcal{AG}$, but, moreover, $\mathcal{AG}$ is an algebraizable institution with $\mathcal{GA}$ its algebraic counterpart. We do this by showing that there exists an interpretation $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle$ from $\mathcal{GA}$ into $\mathcal{AG}$, which is inverse to the interpretation $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle$. Thus, the two institutions $\mathcal{AG}$ and $\mathcal{GA}$ are deductively equivalent institutions, as is required for algebraizability.

To this end, define the natural transformation $\beta : \mathrm{EQ} \to \mathcal{P}\mathrm{SEN}$, by setting, for every set X and all terms $t_0, t_1 \in \mathrm{Tm}(X)$,

$$\beta_X(t_0 \approx t_1) = \{(t_0, t_1)\ I\ (1, 0)\}.$$

Then we have the following analogs of Lemmas 6 and 7 and of Proposition 8 establishing that $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle$ is an interpretation from $\mathcal{GA}$ to $\mathcal{AG}$:

Lemma 10. *Let X be a set, $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$ and $t_0, t_1 \mathrm{Tm}(X)$. Then*

$$\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1 \quad \textit{iff} \quad \langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (t_0, t_1)\ I\ (1, 0).$$

Proof.

$$\begin{array}{rll} \langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1 & \text{iff} & \sigma^*(t_0) = \sigma^*(t_1) \\ & \text{iff} & \sigma^*(t_0) = I(\sigma^*(t_1), 1, 0) \\ & \text{iff} & \sigma^*(t_0) = \sigma^*(T(t_1, 1, 0)) \\ & \text{iff} & \langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (t_0, t_1)\ I\ (1, 0). \end{array}$$

□

Lemma 11. *Let X be a set, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$ and $t_0, t_1 \in \mathrm{Tm}(X)$. Then*

$$\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (1, 0) \quad \textit{iff} \quad \langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_0 \approx t_1.$$

Proof.

$$\begin{array}{rll} \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (1, 0) & \text{iff} & \sigma^*(t_0) = \sigma^*(T(t_1, 1, 0)) \\ & \text{iff} & \sigma^*(t_0) = I(\sigma^*(t_1), 1, 0) \\ & \text{iff} & \sigma^*(t_0) = \sigma^*(t_1) \\ & \text{iff} & \langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_0 \approx t_1. \end{array}$$

□

Proposition 12. *Let X be a set. For all $E \cup \{t_0 \approx t_1\} \subseteq \mathrm{EQ}(X)$,*

$$t_0 \approx t_1 \in C_X^{\mathcal{GA}}(E) \quad \textit{iff} \quad \beta_X(t_0 \approx t_1) \subseteq C_X^{\mathcal{AG}}(\beta_X(E)).$$

Proof. Assume, first, that $t_0 \approx t_1 \in C_X^{\mathcal{GA}}(E)$. This means that, for all $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$, $\langle \mathbf{R}, \sigma\rangle \models_X E$ implies $\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1$. Suppose, now, that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, and $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \beta_X(E)$. Thus, $\langle\langle \mathbf{G}, \mathcal{S}\rangle,\ \sigma\rangle \models_X (\epsilon_0, \epsilon_1)\ I\ (1, 0)$, for all $\epsilon_0 \approx \epsilon_1 \in E$. Therefore, by Lemma 11, $\langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X \epsilon_0 \approx \epsilon_1$, for all $\epsilon_0 \approx \epsilon_1 \in E$. Hence, by hypothesis, we have $\langle \mathsf{R}_{\mathcal{S}}(\mathbf{G}), \sigma\rangle \models_X t_0 \approx t_1$. Again, using Lemma 11, we obtain that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \beta_X(t_0 \approx t_1)$. This proves that $\beta_X(t_0 \approx t_1) \subseteq C_X^{\mathcal{AG}}(\beta_\Sigma(E))$.

Assume, conversely, that $\beta_X(t_0 \approx T_1) \subseteq C_X^{\mathcal{AG}}(\beta_X(E))$. This means that, for all $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \beta_X(E)$ implies $\langle\langle \mathbf{G}, \mathcal{S}\rangle,$

$\sigma\rangle \models_X \beta_X(t_0 \approx t_1)$. Suppose, now, that $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$, such that $\langle \mathbf{R}, \sigma\rangle \models_X E$. Thus, $\langle \mathbf{R}, \sigma\rangle \models_X \epsilon_0 \approx \epsilon_1$, for all $\epsilon_0, \approx \epsilon_1 \in E$. Therefore, by Lemma 10, $\langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X (\epsilon_0, \epsilon_1)\ I\ (1, 0)$, for all $\epsilon_0 \approx \epsilon_1 \in E$. Hence, by hypothesis, we get that $\langle \mathsf{G}(\mathbf{R}), \sigma\rangle \models_X \beta_X(t_0 \approx t_1)$. Again, using Lemma 10, we obtain that $\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1$. This proves that $t_0 \approx t_1 \in C_X^{\mathcal{GA}}(E)$. □

Theorem 13. *The pair $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle$ forms an interpretation from the institution $\mathcal{GA}$ of Hall ternary rings to the institution $\mathcal{AG}$ of abstract geometry.*

Proof. This is simply a restatement of Proposition 12. □

To complete our demonstration that $\mathcal{AG}$ is algebraizable and the institution of Hall ternary rings $\mathcal{GA}$ is its equivalent algebraic semantics, it suffices now to show that the two interpretations $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle : \mathcal{AG} \to \mathcal{GA}$ and $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle : \mathcal{GA} \to \mathcal{AG}$ are inverse of one another in the precise technical sense of [15] (see also [2]). In other words, it must be shown that composing α and β results in interderivable sets of geometric formulas in $\mathcal{AG}$ and composing β and α results in interderivable sets of equations in $\mathcal{GA}$. The following two lemmas pave the way for the final results:

Lemma 14. *Let X be a set, $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$ and*

$$(t_0, t_1)\ I\ (s_0, s_1) \in \mathrm{SEN}(X).$$

Then

$$\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) \quad \textit{iff} \quad \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_1, T(t_0, s_0, s_1))\ I\ (1, 0).$$

Proof.

$$\begin{aligned} \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1) & \\ \text{iff} \quad & \sigma^*(t_1) = \sigma^*(T(t_0, s_0, s_1)) \\ \text{iff} \quad & \sigma^*(T(t_0, s_0, s_1)) = I(\sigma^*(t_1), 1, 0) \\ \text{iff} \quad & \sigma^*(T(t_0, s_0, s_1)) = \sigma^*(T(t_1, 1, 0)) \\ \text{iff} \quad & \langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_1, T(t_0, s_0, s_1))\ I\ (1, 0). \end{aligned}$$

□

Lemma 15. *Let X be a set, $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$ and $t_0, t_1 \in \mathrm{Tm}(X)$. Then*

$$\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1 \quad \textit{iff} \quad \langle \mathbf{R}, \sigma\rangle \models_X t_1 \approx T(t_0, 1, 0).$$

Proof.

$$\begin{aligned}\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1 \quad &\text{iff} \quad \sigma^*(t_0) = \sigma^*(t_1)\\ &\text{iff} \quad \sigma^*(t_1) = I(\sigma^*(t_0), 1, 0)\\ &\text{iff} \quad \sigma^*(t_1) = \sigma^*(T(t_0, 1, 0))\\ &\text{iff} \quad \langle \mathbf{R}, \sigma\rangle \models_X t_1 \approx T(t_0, 1, 0).\end{aligned}$$

□

Theorem 16. *The institution $\mathcal{AG}$ of abstract geometry and the institution $\mathcal{GA}$ of Hall ternary rings are deductively equivalent institutions. Thus, $\mathcal{AG}$ is algebraizable and the institution $\mathcal{GA}$ is its equivalent algebraic semantics.*

Proof. We have already proven in Theorems 9 and 13 that $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle : \mathcal{AG} \to \mathcal{GA}$ and $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle : \mathcal{GA} \to \mathcal{AG}$ are interpretations. Thus, it suffices to show that they are inverse to one another. Suppose, that X is a set and $(t_0, t_1)\ I\ (s_0, s_1) \in \mathrm{SEN}(X)$. Then, for all $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \in |\mathrm{MOD}(X)|$, we have, by Lemma 14, that $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X (t_0, t_1)\ I\ (s_0, s_1)$ iff $\langle\langle \mathbf{G}, \mathcal{S}\rangle, \sigma\rangle \models_X \beta_X(\alpha_X((t_0, t_1)\ I\ (s_0, s_1)))$. Therefore, we get that $C_X^{\mathcal{AG}}((t_0, t_1)\ I\ (s_0, s_1)) = C_X^{\mathcal{AG}}(\beta_X(\alpha_X((t_0, t_1)\ I\ (s_0, s_1))))$. Similarly, if X is a set and $t_0 \approx t_1 \in \mathrm{EQ}(X)$, then, for all $\langle \mathbf{R}, \sigma\rangle \in |\mathrm{ALG}(X)|$, we have, by Lemma 15, that $\langle \mathbf{R}, \sigma\rangle \models_X t_0 \approx t_1$ iff $\langle \mathbf{R}, \sigma\rangle \models_X \alpha_X(\beta_X(t_0 \approx t_1))$. Therefore, we get that $C_X^{\mathcal{GA}}(t_0 \approx t_1) = C_X^{\mathcal{GA}}(\alpha_X(\beta_X(t_0 \approx t_1)))$. This concludes the proof that $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle$ and $\langle \mathrm{I}_{\mathbf{Sign}}, \beta\rangle$ are indeed inverse to one another in the precise technical sense of abstract algebraic logic □

Theorem 16, which is the main theorem of the paper, shows that the institution of abstract (affine plane) geometry is deductively equivalent to the institution of geometric algebra. Thus, the class of all Hall ternary rings forms an equivalent algebraic semantics of affine plane geometry in the precise technical sense of abstract algebraic logic. It is in this sense that one may say that the coordinate rings of modern abstract geometry form an equivalent algebraic semantics of the logic of abstract geometry. Therefore, the process of coordinatization in geometry may be viewed as a special case of the formal process of algebraization of logical systems.

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