Lemma 5.12. *For all formulas $A \in \mathbf{L}(\Box, \Diamond)$ and a valuation V,*

$$\mathcal{R}, V, x \Vdash_{\mathrm{b}} |A| \iff \mathcal{R}, V^{\circ}, x \Vdash_{\mathrm{b}} |A| \iff \mathcal{R}, V^{\circ}, x \Vdash_{\mathrm{r}} A.$$

Proof. By induction on A. □

Theorem 5.13. *Let $A \in \mathbf{L}(\Box, \Diamond)$. Then, A is true in all degenerate intuitionistic relational models if and only if $|A|$ is true in all* (S5 ⊗ K)*-models.*

Proof.

Let us write $\mathcal{R}, V \Vdash_{\mathrm{r}} A$ if $\mathcal{R}, V, x \Vdash_{\mathrm{r}} A$ for all x, and similarly for $\mathcal{R}, V \Vdash_{\mathrm{b}} A$. Then the condition

A is true in all degenerate intuitionistic relational models

is rephrased as

$\mathcal{R}, V \Vdash_{\mathrm{r}} A$ for any $\mathcal{R}$ and admissible V,

where $\mathcal{R}$ ranges over all degenerate intuitionistic relational frames, and V ranges over all $\mathcal{R}$-valuations. This is equivalent to

$\mathcal{R}, V^{\circ} \Vdash_{\mathrm{r}} A$ for any $\mathcal{R}$ and (not necessarily admissible) V,

since V° is admissible for every V, and every admissible valuation V is an admissible variant of some valuation (because $V = V^{\circ}$ if V is admissible). By using the previous lemma, we can rewrite this condition into

$\mathcal{R}, V \Vdash_{\mathrm{b}} |A|$ for any $\mathcal{R}$ and (not necessarily admissible) V,

and this is the same as

$|A|$ is true in all (S5 ⊗ K)-models.

□

6. Concluding Remarks

6.1 Summary

We have investigated semantic aspects of intuitionistic modal logics without distributivity law $\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B$. We have defined neighborhood semantics, and proved that the existing relational semantics can be represented in terms of neighborhood semantics, as well as the converse under a slight restriction. This shows a close relationship between these two semantics.

By using this result, we have also considered the relationship between classical monotone modal logic and normal bimodal logic. We proved that the classical monotone modal logic with $\mathrm{N}_\Diamond$ has relational representation of its neighborhood semantics, and embeddable into $\mathrm{S5} \otimes \mathrm{K}$.

The results obtained from these investigations bring us a new insight in intuitionistic modal logic and classical non-normal modal logic. In particular, it turned out that (some of) the existing intuitionistic modal logic can actually be captured as an intuitionistic version of non-normal modal logic in a natural way.

6.2 Non-Normal Modalities and Multimodal Logics

Translation from non-normal modal logics to normal multimodal logics has already been studied before. Gasquet and Herzig showed that non-normal (not necessarily monotone) modal logic can be translated into normal modal logic with three modalities [7]. Kracht and Wolter proved that monotone modal logic can be "simulated" by normal bimodal logic (actually, they also proved that a normal monomodal logic can simulate monotone modal logics) [10].

The basic idea behind their work is different from ours. Our translation from monotone to bimodal logic is based on the idea of considering the set

$$W^* = \{(x, X) \mid x \in W, X \in N(x)\},$$

which consists of all pairs of possible worlds and their neighborhoods. On the other hand, both of the previous approaches consider the set

$$W \cup \bigcup_{x \in W} N(x),$$

which consists of all possible worlds and all subsets of W that are neighborhoods of some worlds.[2]

This causes the difference in source and target logics of translations. In our translation, both of the source and target are stronger logics than the previous work. We assume the axiom $\mathrm{N}_\Diamond$ in the source logic, and considered $\mathrm{S5} \otimes \mathrm{K}$, a combination of S5 and K, as a target. In the previous work, they did not assume an extra axiom like $\mathrm{N}_\Diamond$, and the target is a combination of two (in the case of Kracht and Wolter) or three (in the case of Gasquet and Herzig) copies of K.

6.3 Relationship with Gödel Translation

Our translation from monotone modal logic to $\mathrm{S5} \otimes \mathrm{K}$ can be considered as a variant of Gödel translation. Wolter and Zakharyaschev [16] investigated an embedding from intuitionistic modal logic into classical normal bimodal logic. They defined a Gödel-style translation, denoted by t, from an intuitionistic modal logic (with $\Box$ as the only primitive modality) into $\mathrm{S4}\otimes\mathrm{K}$. Our translation $|\cdot|$ can be seen as a variant of theirs.

At first sight, there is a difference between these two translations in the case of implication. Wolter and Zakharyaschev's t is defined as

$$t(A \to B) = \Box_1(t(A) \to t(B)),$$

which is the same as the usual Gödel translation, while our version $|\cdot|$ is given by

$$|A \to B| = |A| \to |B|.$$

However, when $\Box_1$ is an S5 modality, this makes no difference; we can prove that $|A| \to |B|$ and $\Box_1(|A| \to |B|)$ are equivalent in $\mathrm{S5} \otimes \mathrm{K}$.

[2] Actually, Kracht and Wolter used more sophisticated technique, but the basic idea is as described here.

6.4 "Internal Observer" Interpretation

One possible interpretation that rejects the distributivity is to consider observers inside possible worlds, which we call "internal observer." Let us assume that an observer o_x is assigned to each possible world x. We will consider a formula A true at a world x if the observer o_x is able to verify that A is true.

$\Diamond(A \vee B)$ at x means that an observer o_x knows $A \vee B$ is true somewhere, say y. Note that this does not necessarily mean that o_x can determine the disjunct that becomes true at y. It is o_y who can know which of A and B is actually true at y. This means that, in view of o_x, neither $\Diamond A$ nor $\Diamond B$ cannot be verified to be true. Therefore $\Diamond(A \vee B)$ does not necessarily imply $\Diamond A \vee \Diamond B$, and this is why the internal observer interpretation rejects distributivity.

In the usual Kripke semantics, unlike this interpretation, we implicitly assume a viewpoint of an observer outside the Kripke frame. So we can say that it takes an external observer's viewpoint. The argument above would not be true if we take this point of view, and the distributivity cannot be rejected (indeed, the usual Kripke semantics admits distributivity).

Similar viewpoint can be found in Aucher's work on internal (and imperfect external) epistemic logic [2]. He investigated an epistemic logic based on the view from inside the situation, rather than the usual view from outside the situation. To model such a circumstance, Aucher considered disjoint sum of several Kripke models.

6.5 Neighborhoods as Ambiguity

The interpretation discussed above is partially expressed in neighborhood semantics. An internal observer o_x does not have complete information about other worlds, so o_x has several possibilities in mind about the actual situation of other worlds. Each of these possible situations is represented as a neighborhood. For example, in the situation above (o_x can verify $\Diamond(A \vee B)$ but not $\Diamond A \vee \Diamond B$), o_x would think of two possibilities about the sets of accessible worlds (which we call X and X'). In X a world that makes A true can be found, and X' contain a world that makes B true. This uncertainty can be expressed in terms of neighborhoods, that is, each neighborhood of x represents a possible set of accessible worlds from x in view of o_x. Incidentally, the decreasing condition on neighborhood function is understood as a natural assumption that the amount of uncertainty would decrease if the amount of knowledge increases.

Another way to express this uncertainty is to consider a pair (x, X), where x is a possible world, and X is a candidate of the set of accessible worlds from x. This construction is precisely what we did to transform a neighborhood model into a relational model. So the notion of possible worlds in relational semantics actually carries two pieces of information, the current state of knowledge and the set of accessible worlds.

A similar idea can be found in Hilken's work [8], which investigates Stone duality for modal frames (a complete Heyting algebras equipped with modal operators). In his theory, the notion of points in modal frames contains two components; a completely prime filter p and an element a of the frame such that $\Diamond a \notin p$.[3] The first component p is the same as the notion of point appearing in the duality theory between frames and spaces. The second component a carries an extra piece of information; intuitively, this is (the interior of) the set of points that are *not* accessible from the point (p, a) represents.

As we have seen above, a kind of uncertainty plays an important role in the semantics of intuitionistic modal logic we have considered in this paper (and the existing literature). This is the source of the difficulty when we try to capture intuitionistic modal logics in the framework of Kripke semantics, which is originally a framework for normal modal logics. What we have presented in this paper is that intuitionistic modal logics can be treated as non-normal modal logics rather than normal ones. We believe this finding advances our understanding of intuitionistic modal logics.

A. Proof of Completeness

Soundness can be checked by the standard induction, so we will prove the completeness only. The basic strategy of the proof is the same as the one found in Section 4 of [3], except that we consider neighborhood frames instead of relational frames.

[3] Precisely speaking, Hilken calls such a pair "pre-point," and defines the set of points as a certain subset of the set of all pre-points.

Lemma A.1 (deduction theorem)**.** *Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic and Γ a Λ-theory. Then, $A \to B \in \Gamma$ if and only if all Λ-theories Δ containing Γ and A contain B.*

Proof. To prove "if" part, let $\Delta = \{C \mid A \to C \in \Gamma\}$. Then Δ is a theory containing Γ and A. Then, from assumption, Δ contains B, and this means $A \to B \in \Gamma$ by definition of Δ.

For the "only if" part, use the fact that any theory is closed under modus ponens. □

Definition A.2. Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic and Γ a Λ-theory. Γ is said to be *prime* if it satisfies the following conditions:

1. if $A \vee B \in \Gamma$, then either $A \in \Gamma$ or $B \in \Gamma$;
2. $\bot \notin \Gamma$.

Lemma A.3 (extension lemma)**.** *Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic and Γ a Λ-theory not containing A. Then, there exists a prime Λ-theory Δ such that $A \notin \Delta$ and $\Gamma \subseteq \Delta$.*

Proof. A maximal element of $\{\Delta \mid \Gamma \subseteq \Delta, A \notin \Delta\}$ has the required property. □

Definition A.4. Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic and Γ a Λ-theory. Then we define $\Box^{-1}\Gamma := \{A \mid \Box A \in \Gamma\}$.

Lemma A.5. *Let Λ be an $\mathbf{L}(\Box, \Diamond)$-logic containing axiom K. If Γ is a Λ-theory, then so is $\Box^{-1}\Gamma$.*

Proof. If $A \to B \in \Box^{-1}\Gamma$ and $A \in \Box^{-1}\Gamma$, then $\Box(A \to B) \in \Gamma$ and $\Box A \in \Gamma$. Since Λ contains K, it follows that $\Box B \in \Gamma$, hence $B \in \Box^{-1}\Gamma$. So $\Box^{-1}\Gamma$ is closed under modus ponens. □

Below, we will construct a canonical model for a logic Λ and prove standard properties. In what follows, we fix an arbitrary $\mathbf{L}(\Box, \Diamond)$-logic Λ containing $\mathrm{IM} + \mathrm{N}_{\Diamond\Box}$.

Definition A.6 (canonical neighborhood frame)**.** We define $\mathcal{N}^{\Lambda}$ to be a tuple $\langle W^{\Lambda}, \leq^{\Lambda}, N^{\Lambda}\rangle$, where

- W^{Λ} is the set of all prime Λ-theories,

- $\leq^\Lambda$ is the inclusion relation between sets, and
- N^Λ is defined by

$$N^\Lambda(\Gamma) = \left\{ n(\Gamma', A) \,\middle|\, \Gamma \subseteq \Gamma' \in W^\Lambda, \Diamond A \notin \Gamma' \right\},$$
$$n(\Gamma, A) = \left\{ \Delta \in W^\Lambda \,\middle|\, \Box^{-1}\Gamma \subseteq \Delta \text{ and } A \notin \Delta \right\}$$

for each $\Gamma \in W^\Lambda$.

Definition A.7 (canonical valuation)**.** A valuation V^Λ on $\mathcal{N}^\Lambda$ is defined by

$$V^\Lambda(p) = \{\Gamma \mid p \in \Gamma\}.$$

Definition A.8 (canonical model)**.** Clearly V^Λ is admissible, so $\mathcal{M}^\Lambda = \langle \mathcal{N}^\Lambda, V^\Lambda \rangle$ is an intuitionistic neighborhood model. We call this model a *canonical model for* Λ.

As usual, the following holds.

Theorem A.9. $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} A$ *if and only if* $A \in \Gamma$.

Proof. We proceed by induction on A. The cases when A is an atomic formula, $\bot$, conjunction, and disjunction are trivial.

$\underline{A \to B \in \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} A \to B}$: For all $\Delta \geq^\Lambda \Gamma$, if $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_{\mathrm{n}} A$, then $A \in \Delta$ by induction hypothesis, so $A, A \to B \in \Delta$. Since Δ is closed under modus ponens, we obtain $B \in \Delta$. This means that $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} B$ by induction hypothesis.

$\underline{A \to B \notin \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} A \to B}$: Suppose $A \to B \notin \Gamma$. Then, by deduction theorem and extension lemma, there exists $\Delta \in W^\Lambda$ such that $A \in \Delta$, $\Gamma \subseteq \Delta$, and $B \notin \Delta$. So $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} A \to B$.

$\underline{\Box A \in \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} \Box A}$: If $\Delta \in n(\Gamma', B) \in N^\Lambda(\Gamma)$ for some Γ' and B, then $\Box^{-1}\Gamma \subseteq \Box^{-1}\Gamma' \subseteq \Delta$. Since $\Box A \in \Gamma$, we have $A \in \Box^{-1}\Gamma \subseteq \Delta$. Therefore $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_{\mathrm{n}} A$. Since this holds for all Γ', B and Δ, it follows that $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_{\mathrm{n}} \Box A$.

$\underline{\Box A \notin \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_{\mathrm{n}} \Box A}$: First, note that $\Box A \notin \Gamma$ means $\Diamond\bot \notin \Gamma$, since

$$\Diamond\bot \to \Box\bot, \Box\bot \to \Box A \in \Gamma$$

from $N_{\Diamond\Box}$ and the monotonicity of $\Box$. This means that $n(\Gamma, \bot) \in N^\Lambda(\Gamma)$. So it suffices to show that $n(\Gamma, \bot)$ contains some Δ such that $A \notin \Delta$. Such Δ can be obtained as follows. Since $\Box A \notin \Gamma$, we have $A \notin \Box^{-1}\Gamma$. By using extension lemma we can obtain a prime Λ-theory $\Delta \supseteq \Box^{-1}\Gamma$ with $A \notin \Delta$. For such Δ, it holds that $\Delta \in n(\Gamma, \bot)$.

$\underline{\Diamond A \in \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_n \Diamond A}$: Take an arbitrary $n(\Gamma', B) \in N^\Lambda(\Gamma)$. Then we have $\Diamond B \notin \Gamma'$. Let Θ be the least theory containing $\Box^{-1}\Gamma'$ and A.

We first show that $B \notin \Theta$. If $B \in \Theta$, then we would have $A \to B \in \Box^{-1}\Gamma'$ from deduction theorem. This means $\Box(A \to B) \in \Gamma'$, hence $\Diamond A \to \Diamond B \in \Gamma'$ since

$$\Box(A \to B) \to \Diamond A \to \Diamond B \in \Gamma'.$$

However, $\Diamond A \in \Gamma \subseteq \Gamma'$ from assumption, so it follows that $\Diamond B \in \Gamma'$, a contradiction.

Now we have $\Box^{-1}\Gamma' \subseteq \Theta$, $A \in \Theta$, and $B \notin \Theta$. By extension lemma, there exists Δ satisfying the same conditions. For such Δ, we have $\Delta \in n(\Gamma', B)$, and $A \in \Delta$, and hence $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_n A$.

So we have proved that for all neighborhood $n(\Gamma', B)$ of Γ there exists $\Delta \in n(\Gamma', B)$ such that $\mathcal{N}^\Lambda, V^\Lambda, \Delta \Vdash_n A$. This means that $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \Vdash_n \Diamond A$.

$\underline{\Diamond A \notin \Gamma \implies \mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_n \Diamond A}$: Assume $\Diamond A \notin \Gamma$, and let $X = n(\Gamma, A)$. Clearly X is a neighborhood of Γ, and any of its element Δ does not contain A. This means that $\mathcal{N}^\Lambda, V^\Lambda, \Delta \nVdash_n A$ for all $\Delta \in X$. Therefore $\mathcal{N}^\Lambda, V^\Lambda, \Gamma \nVdash_n \Diamond A$.

□

The following is an easy consequence of this theorem.

Lemma A.10. *Let $\mathcal{K}$ be a class of intuitionistic neighborhood models. If $\mathcal{M}^\Lambda \in \mathcal{K}$, then Λ is complete with respect to $\mathcal{K}$, that is, any formula true in $\mathcal{K}$ is a theorem of Λ.*

By using this lemma, the completeness parts of Theorem 3.6 and Theorem 3.9 can be reduced to the following lemma, which is easily verified.

Lemma A.11. *1. The canonical model of* $\mathrm{IM}+\mathrm{N}_{\Diamond\Box}$ *is an intuitionistic neighborhood model.*

2. The canonical model of $\mathrm{IM}+\mathrm{N}_{\Diamond}$ *is a normal intuitionistic neighborhood model.*

Proof. The first claim is immediate from the definition of N^{Λ}. For the second part, it suffices to show that $N^{\mathrm{IM}+\mathrm{N}_{\Diamond}}(\Gamma) \neq \emptyset$ for each Γ. Actually, $n(\Gamma, \bot)$ is always a neighborhood of Γ. This follows from the presence of $\mathrm{N}_{\Diamond}$: any prime Γ does not contain $\Diamond\bot$ since $\neg\Diamond\bot \in \Gamma$ and $\bot \notin \Gamma$. □

Next, we will prove Theorem 5.7. Here, we identify a classical neighborhood frame $\langle W, N\rangle$ and an intuitionistic neighborhood frame $\langle W, =, N\rangle$, and similarly for a classical neighborhood model. The following is also immediate from Lemma A.10.

Lemma A.12. *A formula* $A \in \mathbf{L}(\Box, \Diamond)$ *is a theorem of* $\mathrm{IM}+\mathrm{PEM}+\mathrm{N}_{\Diamond}$ *if and only if it is true in all normal classical neighborhood models.*

Proof. It is straightforward to verify that the canonical model of $\mathrm{IM} + \mathrm{PEM} + \mathrm{N}_{\Diamond}$ is a normal classical neighborhood model. Therefore, completeness follows from Lemma A.10. Soundness is proved in the usual way. □

So it suffices to prove that degenerate intuitionistic relational models and normal classical neighborhood models determine the same logic. Basically this is done by mutual translations between models presented in Section 4, but there is a subtle problem. If $\langle \mathcal{N}, V\rangle$ is a normal classical neighborhood model, then its translation is a degenerate intuitionistic relational model, so this direction is straightforward. Consider the other direction. If we have a degenerate intuitionistic relational model $\langle \mathcal{R}, V\rangle$, by translation we obtain a model $\langle \mathcal{N}_{\mathcal{R}}, V\rangle$, which is not necessarily classical. A neighborhood frame is classical when its $\leq$-part is the equality $=$, but in this case this is not the case (it is only an equivalence relation).

Actually, this is not a big problem. We can fix this by considering quotient of $\mathcal{N}_{\mathcal{R}}$, which is indeed a classical neighborhood frame. In general, we can prove the following:

Proposition A.13. *Let* $\mathcal{N} = \langle W, \leq, N\rangle$ *be an intuitionistic neighborhood frame, and* V *an admissible* $\mathcal{N}$*-valuation. Define its quotient* $|\mathcal{N}| = \langle |W|, |\leq|, |N|\rangle$ *and* $|V|$ *as follows.*

- *$|W| = W/{\sim}$, where $x \sim y$ if and only if $x \le y$ and $y \le x$.*
- *$[x] \; |{\le}| \; [y]$ if and only if $x \le y$, where $[z]$ denotes the equivalence class of z. This does not depend on the choice of x and y.*
- *$|N|([x]) = \{X/{\sim} \mid X \in N(x)\}$, where $X/{\sim}$ is the image of X under the canonical projection $W \to |W|$. Since N is decreasing, $x \sim y$ implies $N(x) = N(y)$, so $|N|$ is well-defined.*
- *$|V|(p) = V(p)/{\sim}$.*

Then, for any $A \in \mathbf{L}(\Box, \Diamond)$, we have

$$\mathcal{N}, V, x \Vdash_{\mathrm{n}} A \iff |\mathcal{N}|, |V|, [x] \Vdash_{\mathrm{n}} A.$$

Proof. By induction on A. $\Box$

For any intuitionistic neighborhood frame $\mathcal{N} = \langle W, \le, N \rangle$, preorder structure $|{\le}|$ of its quotient $|\mathcal{N}|$ is clearly an order, that is, it is antisymmetric. In particular, when $\le$ is an equivalence relation, $|{\le}|$ is the equality on the quotient $|W|$. Applying this construction to $\mathcal{N}_{\mathcal{R}}$, we can check that each degenerate intuitionistic relational model has an equivalent normal classical neighborhood model $|\mathcal{N}|_{\mathcal{R}}$. This completes the proof of Theorem 5.7.

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