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INCLUSIONS BETWEEN PSEUDO-EUCLIDEAN MODAL LOGICS

A b s t r a c t. We describe properties of simply axiomatized modal logics, which are called pseudo-Euclidean modal logics. For fixed non-negative integers m and n, let $\mathbf{E}_k^{m,n}$ be the logic which is obtained from the smallest normal propositional modal logic $\mathbf{K}$ by adding the pseudo-Euclidean axiom $\Diamond^k \phi \to \Box^m \Diamond^n \phi$, where $k \geq 0$. We will then give a complete description of the inclusion relationship among these logics by showing inclusion relationships for pairs of their logics with fixed m and n.

1. Introduction

One of the simplest kinds of modal axioms are modal reduction principles (MRP) first introduced by Fitch (1973) [5], and further studied by different authors. Probably, the two most striking results on MRP are the following: the non-elementarity of $\mathbf{K} + \Box\Diamond\phi \to \Diamond\Box\phi$ (Van Benthem - Goldblatt, 1975) [1] [6] and the finite model property of uniform modal logics (Fine,

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1975) [4]. Among many natural properties of MRP-logics, only elementarity is completely investigated for the monomodal case by Van Benthem; the polymodal case remains unclear. The situation with other properties is much worse. Very little is known on the finite model property of non-uniform logics and nothing is known on completeness of non-uniform logics beyond Sahlqvist's theorem. The works by Chagrov-Shehtman (1995) [2] and Kracht (1999) [8] give examples of undecidable polymodal and temporal MRP-logics; the proofs are based on encoding the word problem for semigroups. This technique can also be used to show that inclusion between finitely axiomatizable polymodal MRP-logics is undecidable. But the same problem for the monomodal case remains a big challenge.

Inclusion relationships among various propositional modal logics have been found since the early work on modal logics. For example, the inclusion relationship among a class of logics above **K45** is shown in [9]. Our work throws light on the proof theoretical strength of logical systems among pseudo-Euclidean modal logics.

Throughout this paper, m and n are fixed non-negative integers. Let $\mathbf{E}_k^{m,n}$ be the logic which is obtained from the smallest normal modal logic $\mathbf{K}$ by adding the axiom $\Diamond^k\phi \to \Box^m\Diamond^n\phi$, where $k \geq 0$. Here, $\Diamond^k\phi$ and $\Box^{k'}\phi$ denote formulas $\Diamond\cdots\Diamond\phi$ with k diamonds and $\Box\cdots\Box\phi$ with k' boxes, respectively. We call any logic of the form $\mathbf{E}_k^{m,n}$, a *pseudo-Euclidean* modal logic. Since each axiom $\Diamond^k\phi \to \Box^m\Diamond^n\phi$ is a Sahlqvist formula, we can show that the logic $\mathbf{E}_k^{m,n}$ is Kripke complete for each k. In fact, let us say that a binary relation R on a set W is (k,m,n)*-pseudo-Euclidean* if for any $x, y, z \in W$, xR^ky and xR^mz imply zR^ny. Then, it is easy to see that $\mathbf{E}_k^{m,n}$ is Kripke complete with respect to the class of all Kripke frames of the form (W, R) with a (k, m, n)-pseudo-Euclidean relation R on W. Note that R is $(1, 1, 1)$-pseudo-Euclidean if and only if it is Euclidean. Let $\mathcal{PE}_k^{m,n}$ be the class of all Kripke frames of the form (W, R), where R is a (k, m, n)-pseudo-Euclidean relation on W. Then it is easy to see that $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ if and only if $\mathcal{PE}_k^{m,n} \subseteq \mathcal{PE}_{k'}^{m,n}$. In the rest of this paper, we identify the axiom system $\mathbf{E}_k^{m,n}$ with the set of all formulas provable in $\mathbf{E}_k^{m,n}$. Our main goal in this paper is to show when $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ holds. Note that $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ trivially holds when $k = k'$. So, we assume $k \neq k'$ in the following. Our result is summarized in the following theorem where we use " | " to mean that $x \mid y$ if and only if y is divisible by x.

Theorem 1. *1. For $k > k'$: $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ iff $m = 0$ and $k' = n$.*

2. For $k' > k$: *2a.* *If $m = 0$ and $n = k'$ then $\mathbf{E}^{m,n}_k \supseteq \mathbf{E}^{m,n}_{k'}$.*

2b. *Suppose that either $m > 0$ or $n \neq k'$.*
If one of the following (1), (2), (3) holds
(1) $k \geq m + n$,
(2) $m \geq k$ and $m > n$,
(3) $m = n \geq k > 0$,
then
$\mathbf{E}^{m,n}_k \supseteq \mathbf{E}^{m,n}_{k'}$ *iff* $(k - m - n) \mid (k' - m - n)$.

2c. *Otherwise, $\mathbf{E}^{m,n}_k \not\supseteq \mathbf{E}^{m,n}_{k'}$.*

A detailed proof of our theorem is given also in Ph.D. thesis [7] written by the first author.

2. Proof of the theorem

The rest of the paper will be devoted to an outline of the proof of Theorem 1. It is obvious that $\mathbf{E}^{m,n}_k = \mathbf{E}^{m,n}_{k'}$ when $k = k'$. Henceforth, we assume $k \neq k'$. Also, when $m = 0$ and $k' = n$, the axiom $\Diamond^{k'}\phi \to \Box^m \Diamond^n \phi$ becomes $\Diamond^n\phi \to \Diamond^n\phi$, which is obviously provable in $\mathbf{K}$. That is, $\mathbf{E}^{0,n}_n$ coincides with $\mathbf{K}$. Hence, we have the following.

Lemma 2. *If $m = 0$ and $n = k'$ then $\mathbf{E}^{m,n}_k \supseteq \mathbf{E}^{m,n}_{k'} = \mathbf{K}$.*

When $k > k'$, the converse of Lemma 2 holds as shown below.

Lemma 3. *If $k > k'$ and either $m > 0$ or $n \neq k'$ then $\mathbf{E}^{m,n}_k \not\supseteq \mathbf{E}^{m,n}_{k'}$.*

Proof. Suppose first that $k > m$. We define a frame $\mathcal{F} = (W, R)$ as follows: $W = \{w_i \mid 0 \leq i \leq k' + m\}$, and the binary relation R is defined by 1) $w_i R w_{i-1}$ for each $i = 1, \cdots, m$, and 2) $w_i R w_{i+1}$ for each $i = m, m+1, \cdots, k' + m - 1$.

Then, we can show that both $w_m R^{k'} w_{k'+m}$ and $w_m R^m w_0$ hold, while $w_0 R^n w_{k'+m}$ doesn't, since either $m > 0$ or $k' \neq n$. Thus, if $w_i \models \phi$ only for $i = k' + m$ then $w_m \not\models \mathbf{E}^{m,n}_{k'}$. Therefore $\mathcal{F} \notin \mathcal{PE}^{m,n}_{k'}$. On the other hand, for each $x \in W$, there is no $y \in W$ such that $xR^k y$ since $k > k'$ and $k > m$. Therefore $\mathcal{F} \in \mathcal{PE}^{m,n}_k$.

Suppose next that $k \leq m$. Let us take a frame $\mathcal{G} = (V, S)$ defined as follows: $V = \{w_i \mid 0 \leq i \leq k' + 1\}$, and the binary relation S is defined by

1) $w_0 S w_0$, 2) $w_1 S w_0$, and 3) $w_i S w_{i+1}$ for each $i = 1, \cdots, k'$. Similar to the above, we can show that $\mathcal{G} \in \mathcal{PE}_k^{m,n}$ but $\mathcal{G} \notin \mathcal{PE}_{k'}^{m,n}$. □

Thus, we have proved the first part of Theorem 1. The following lemma holds for arbitrary k and k'.

Lemma 4. *If* $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ *then* $(k-m-n) \,\Big|\, (k'-m-n)$.

Proof. Suppose that $k \neq m+n$ and $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ but $(k-m-n) \,\Big|\, (k'-m-n)$ doesn't hold. Let $a = k-m-n$ and define a frame $\mathcal{F} = (W, R)$ as follows: $W = \{w_i \mid 0 \leq i \leq a-1\}$, and $w_i R w_j$ iff $j \equiv i+1 \pmod a$.

By the assumption, since $k'-m \neq n + h(k-m-n)$ for any $h \in \boldsymbol{Z}$, i.e. $k'-m \not\equiv n \pmod a$, we do not have $w_m R^n w_{k'}$. On the other hand, both $w_0 R^{k'} w_{k'}$ and $w_0 R^m w_m$ hold. Thus $\mathcal{F} \notin \mathcal{PE}_{k'}^{m,n}$. Next, suppose that $w_i R^k w_j$ and $w_i R^m w_s$. Then, $j - i \equiv k \pmod a$ and $s - i \equiv m \pmod a$. Hence $j - s \equiv k - m \pmod a$. But $k - m \equiv n \pmod a$ since $a = k-m-n$. Thus $j - s \equiv n \pmod a$, i.e. $w_s R^n w_j$. Hence $\mathcal{F} \in \mathcal{PE}_k^{m,n}$. This contradicts our assumption that $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$.

Suppose that $k = m+n$. Let $b = \max(k', m)$ and define a frame $\mathcal{G} = (V, S)$ as follows: $V = \{w_i \mid 0 \leq i \leq b\}$, and $w_i S w_{i+1}$ for each $i = 0, \cdots, b-1$. Then $\mathcal{G} \in \mathcal{PE}_k^{m,n}$ holds since $k = m+n$. In this frame, both $w_0 S^{k'} w_{k'}$ and $w_0 S^m w_m$ hold. But we do not have $w_m S^n w_{k'}$ since $k'-m \neq n$. Hence $\mathcal{G} \notin \mathcal{PE}_k^{m,n}$. Thus we have $\mathbf{E}_k^{m,n} \not\supseteq \mathbf{E}_{k'}^{m,n}$. □

In the following, we will find sufficient conditions by which the converse of Lemma 4 holds. We can assume that $k' > k$, and moreover that either $m > 0$ or $n \neq k'$, by Lemma 2.

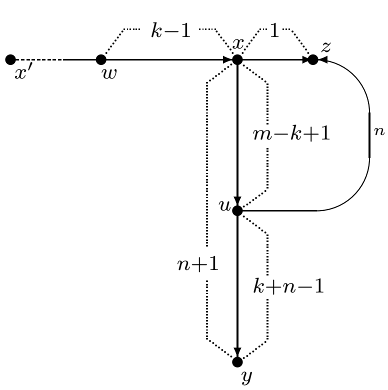
Lemma 5. *If* $k' > k \geq m+n$ *and* $(k-m-n) \,\Big|\, (k'-m-n)$ *then* $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$.

Proof. By the assumption, $k'-m-n = h(k-m-n)$, that is $k' = k + (h-1)(k-m-n)$, for a certain number $h \in \boldsymbol{Z}$. Since $k' > k$ and $k-m-n \geq 0$, we can assume that $k' = k + (h-1)(k-m-n)$ with $h > 1$. To show that $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n} = \mathbf{E}_{k+(h-1)(k-m-n)}^{m,n}$, it is enough to show that every $(W, R) \in \mathcal{PE}_k^{m,n}$ belongs also to $\mathcal{PE}_{k+(h-1)(k-m-n)}^{m,n}$ for any $h > 1$. This can be shown by induction on h.

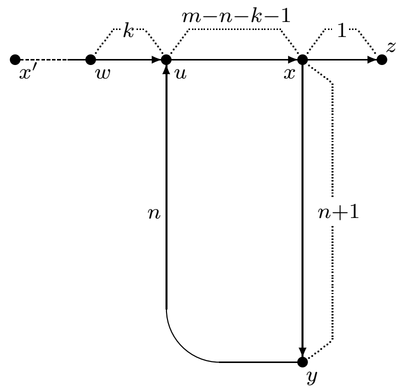
The base step, that is the case of $h = 2$, can be shown in a way similar to the induction step. So, we assume that this holds for h. To show that (W, R) belongs to $\mathcal{PE}_{k+h(k-m-n)}^{m,n}$, we assume that $x R^{k+h(k-m-n)} y$ and $x R^m z$. Then, for some $w \in W$, both $xR^{k+(h-1)(k-m-n)}w$ and $wR^{k-m-n}y$ hold, since $k + (h-1)(k-m-n) \geq 0$ and $k-m-n \geq 0$. Since (W,R) belongs to $\mathcal{PE}^{m,n}_{k+(h-1)(k-m-n)}$ the induction hypothesis gives $xR^{k+(h-1)(k-m-n)}w$ and $xR^m z$ imply $zR^n w$. Since $xR^m z$, $zR^n w$ and $wR^{k-m-n}y$ hold, we have $xR^k y$. But since (W,R) is in $\mathcal{PE}^{m,n}_k$, we also have $zR^n y$. Thus, we have shown that (W,R) belongs to $\mathcal{PE}^{m,n}_{k+h(k-m-n)}$. □

Lemma 6. *Suppose that $m \geq k$ and either 1) $m > n$ or 2) $m = n$ and $k > 0$. Let (W,R) be in $\mathcal{PE}^{m,n}_k$. Then for any $l \geq 0$ and any $M \geq \max(m-n-1, k-1)$, if $xR^{n+l}y$, $xR^l z$ and $x'R^M x$ then $zR^n y$.*

Proof. We will proceed by induction on l. If $l = 0$, this is trivial. When $l = 1$, we will divide the case into two. First, suppose that $k \geq m-n$. Then, for some $w, u \in W$, each of $x'R^{M-(k-1)}w$, $wR^{k-1}x$, $xR^{m-k+1}u$ and $uR^{k+n-m}y$ hold, since $M \geq k-1 \geq 0$, $m-k+1 > 0$ and $k+n-m \geq 0$. Since $wR^{k-1}x$ and xRz hold, we have $wR^k z$. Also, since $wR^{k-1}x$ and $xR^{m-k+1}u$ hold, we have $wR^m u$. Since (W,R) is in $\mathcal{PE}^{m,n}_k$, we have $uR^n z$. Then, for some $v \in W$, we have $x'R^{M+m-k+1-(m-n)}v$ and $vR^{m-n}u$, since $M+m-k+1-(m-n) \geq 0$ and $m-n \geq 0$. Since $vR^{m-n}u$ and $uR^{k+n-m}y$ hold, we have $vR^k y$. Also, since $vR^{m-n}u$ and $uR^n z$ hold, we have $vR^m z$. Therefore $zR^n y$ since (W,R) is in $\mathcal{PE}^{m,n}_k$.



(a) $k \geq m-n$



(b) $k < m-n$

Figure 1:

When $k < m-n$, for some $w, u \in W$, we must have $x'R^{M-k-(m-n-k-1)}w$, $wR^k u$ and $uR^{m-n-k-1}x$, since $M \geq k + (m-n-k-1)$, $k \geq 0$ and $m-n-k-1 \geq 0$. Since $wR^k u$, $uR^{m-n-k-1}x$ and $xR^{n+1}y$ hold, we have $wR^m y$. Since (W,R) is in $\mathcal{PE}^{m,n}_k$, we have $yR^n u$. Then, for some $v \in W$, we must have $wR^{m-k}v$ and $vR^k y$, since $m-k \geq 0$ and $k \geq 0$. Since $vR^k y$, $yR^n u$, $uR^{m-n-k-1}x$ and xRz hold, we have $vR^m z$. Since (W,R) is in $\mathcal{PE}_k^{m,n}$, we have $zR^n y$. Therefore, we have shown the lemma for $l=1$.

Now, for the induction step, we assume that this holds for some $l \geq 1$. To show the lemma for $l+1$, we assume that $xR^{n+l+1}y$, $xR^{l+1}z$ and $x'R^M x$. Then, for some $y', z' \in W$, we must have $xR^{n+1}y'$, $y'R^l y$, xRz' and $z'R^l z$. Hence $z'R^n y'$ by the result when $l=1$. Since $z'R^n y'$ and $y'R^l y$ hold, we have $z'R^{n+l}y$. Since $x'R^M x$ and xRz' hold, we have $x'R^{M+1}z'$. Since $z'R^{n+l}y$, $z'R^l z$, $x'R^{M+1}z'$ and $M+1 \geq M \geq \max(m-n-1, k-1)$, we have $zR^n y$ by the induction hypothesis. □

Lemma 7. *Suppose $k' > k$ and $m \geq k$. Moreover suppose that either 1) $m > n$ or 2) $m = n$ and $k > 0$. Then $(k-m-n) \Big| (k'-m-n)$ implies $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$.*

Proof. By the assumption, $k'-m-n = h(m+n-k)$, that is $k' = k+(h+1)(m+n-k)$, for a certain number $h \in \boldsymbol{Z}$. Since $k' > k$ and $m+n-k \geq 0$, we can assume that $k' = k+(h+1)(m+n-k)$ with $h \geq 0$. To show that $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n} = \mathbf{E}_{k+(h+1)(m+n-k)}^{m,n}$, it is enough to show that every $(W,R) \in \mathcal{PE}_k^{m,n}$ belongs also to $\mathcal{PE}_{k+(h+1)(m+n-k)}^{m,n}$ for any $h \geq 0$. This can be shown by induction on h.

If $h=0$ then $k' = m+n$. we assume that $(W,R) \in \mathcal{PE}_k^{m,n}$, and also that $xR^{m+n}y$ and $xR^m z$ for $x,y,z \in W$. Then, for some $w \in W$, we must have $xR^k w$ and $wR^{m+n-k}y$, since $m+n-k \geq 0$. Then $zR^n w$ and $wR^{m+n-k}y$ by the assumption, so $zR^{m-k+2n}y$. Then, for some $u, v \in W$, we must have $xR^{m-k}u$, $uR^k z$, $zR^{m-k}v$ and $vR^{2n}y$, since $m-k \geq 0$. Since $uR^m v$ and $uR^k z$, we obtain $vR^n z$. But by using Lemma 6, we also have $zR^n y$ by taking $l=n$. Hence $(W,R) \in \mathcal{PE}_{m+n}^{m,n}$.

Since the essence of the proof is involved in the base step, we can check the induction step in a way similar to the base step. □

Thus, combining Lemma 7 with Lemma 4 and 5 we have the following.

Corollary 8. *Suppose that $k' > k$ and that either $m > 0$ or $k' \neq n$. If one of the following (1), (2), (3) holds*

(1) $k \geq m+n$

(2) $m \geq k$ and $m > n$

(3) $m \geq k$, $m = n$ and $k > 0$

then $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ iff $(k-m-n) \mid (k'-m-n)$.

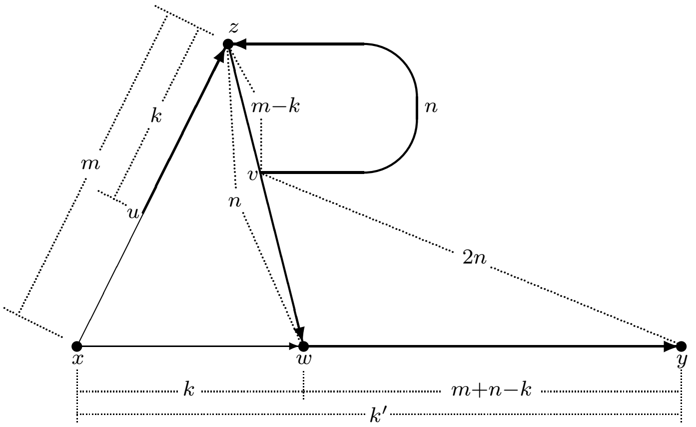


Figure 2:

Finally, we will show that $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ never hold in the remaining cases. So, we assume that none of (1), (2) and (3) in the above corollary holds.

First, suppose that $m > 0$. Suppose moreover that $k > m$. Note that $m + n > k$ holds, because (1) of Corollary 8 doesn't hold.

Lemma 9. *If $k' > k$ and $m + n \geq k > m > 0$ then $\mathbf{E}_k^{m,n} \not\supseteq \mathbf{E}_{k'}^{m,n}$.*

Proof. Define a frame $\mathcal{F} = (W, R)$ as follows: $W = \{w_i \mid 0 \leq i \leq m + n + 1\}$, and 1) $w_i R w_i$ for each $i = m + 1, \cdots, m + n + 1$; 2) $w_i R w_{i+1}$ for each $i = 0, \cdots, m + n$; 3) $w_i R w_{i-1}$ for each $i = m + 2, \cdots, m + n + 1$; 4) $w_0 R w_{m+n+1-k}$.

First, we will show that $\mathcal{F} \in \mathcal{PE}_k^{m,n}$. If $i \geq 1$, $w_i R^k w_j$ and $w_i R^m w_{j'}$ then both w_j and $w_{j'}$ are between w_{m+1} and w_{m+n+1} since $i + k \geq m + 1$ and $i + m \geq m + 1$. Thus $w_{j'} R^n w_j$. If $w_0 R^k w_j$ and $w_0 R^m w_{j'}$ then $w_{j'} R^n w_j$ since $m + 1 \leq j \leq m + n$ and $m \leq j' < m + n$. Hence $\mathcal{F} \in \mathcal{PE}_k^{m,n}$. On the other hand, $w_m R^n w_{m+n+1}$ doesn't hold since $m \neq 0$, while both $w_0 R^{k'} w_{m+n+1}$ and $w_0 R^m w_m$ hold. (Note here that $w_0 R^{k+1} w_{m+n+1}$ and $k + 1 \leq k'$.) Hence $\mathcal{F} \notin \mathcal{PE}_{k'}^{m,n}$. □

Suppose next that $m \geq k$. Because requirement (2) of Corollary 8 doesn't hold, we know that $n \geq m$. We assume first that $n > m > 0$. Then we have the following.

Lemma 10. *If $k' > k$, $m \geq k \geq 0$ and $n > m > 0$ then $\mathbf{E}_k^{m,n} \not\supseteq \mathbf{E}_{k'}^{m,n}$.*

Proof. If $k' < m+n$ then $m+n-k > m+n-k' > 0$, so $(k-m-n) \mid (k'-m-n)$ doesn't hold. Thus, we can derive our conclusion by using Lemma 4. It is therefore sufficient to consider the case where $k' \geq m+n$. We will divide the case into two.

For $n \geq k+m$, we define a frame $\mathcal{F} = (W, R)$ as follows: $W = \{w_i \mid 0 \leq i \leq m+n\}$, and $w_i R w_j$ iff $|i-j| \leq 1$. Since $m+n > n$ by $m > 0$, $w_0 R^n w_{m+n}$ doesn't hold while both $w_0 R^m w_0$ and $w_0 R^{k'} w_{m+n}$ hold for $k' \geq m+n$. Therefore $\mathcal{F} \notin \mathcal{PE}^{m,n}_{k'}$. We will next show that $\mathcal{F} \in \mathcal{PE}^{m,n}_{k}$. We first note that $w_i R^t w_j$ holds if and only if $|i-j| \leq t$. Now, suppose that $w_i R^k w_j$ and $w_i R^m w_s$. Then, $|i-j| \leq k$ and $|i-s| \leq m$. Therefore, $|s-j| \leq |s-i| + |i-j| \leq m+k \leq n$. Hence, $w_s R^n w_j$.

For $n < k+m$, define a frame $\mathcal{G} = (V, S)$ as follows (see Figure 3): $V = \{v_i \mid 0 \leq i \leq k+m+1\}$, and

$v_i S v_j \Leftrightarrow$ either

1) $|i-j| \leq 1$ if $0 \leq i, j \leq k+m+1$ or
2) $j = k+m-n+2$ if $1 \leq i < k+m-n+2$ or
3) $j = n-1$ if $n-1 \leq i \leq k+m$.

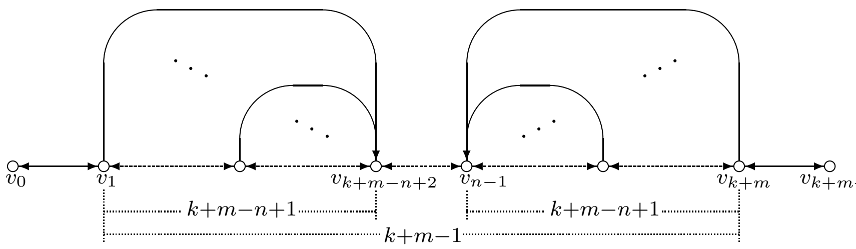


Figure 3:

Note that the frame takes at least $n+1$ steps from v_0 to v_{k+m+1} by the relation S. Thus $v_0 S^n v_{k+m+1}$ doesn't hold. But both $v_m S^{k'} v_{k+m+1}$ and $v_m S^m v_0$ hold because of $k+m+1 \leq k'+m$. Thus $\mathcal{G} \notin \mathcal{PE}^{m,n}_{k'}$.

Assume that $xS^k y$ and $xS^m z$ for any $x, y, z \in V$. Then both y and z must be either between v_0 and v_{k+m}, or between v_1 and v_{k+m+1}, depending on x. For each case, y is accessible from z by n steps, i.e. $zS^n y$. Therefore $\mathcal{G} \in \mathcal{PE}^{m,n}_{k}$. □

Next, assume that $n = m > 0$. Since requirement (3) on Corollary 8 doesn't hold, k must be equal to 0. Then, we have the following.

Lemma 11. *If $k' > k$, $m = n > 0$ and $k = 0$ then* $\mathbf{E}^{m,n}_{k} \not\supseteq \mathbf{E}^{m,n}_{k'}$.

Proof. We define a frame $\mathcal{F} = (W, R)$ as follows;

$$\begin{aligned} W &= \{w_i \mid 0 \leq i \leq m+1\}, \\ w_i R w_j &\Leftrightarrow |i-j| \leq 1. \end{aligned}$$

Then $w_0 R^n w_{m+1}$ doesn't hold while both $w_1 R^{k'} w_0$ and $w_1 R^m w_{m+1}$ hold. Hence $\mathcal{F} \notin \mathcal{PE}_{k'}^{m,n}$. On the other hand, $xR^m y$ implies $yR^n x$ since the frame R is symmetric. Thus $\mathcal{F} \in \mathcal{PE}_0^{m,n}$. □

Finally suppose that $m = 0$. Then by our assumption, we have $n \neq k'$. Since the condition (1) $k \geq m + n = n$ on Corollary 8 doesn't hold, we have $n > k$. Then, we have the following.

Lemma 12. *If $k' > k$, $m = 0$, $n > k$ and $k' \neq n$ then $\mathbf{E}_k^{m,n} \not\supseteq \mathbf{E}_{k'}^{m,n}$.*

Proof. Similarly to Lemma 10, we can show our lemma easily when $k' < n$. So, suppose that $k' > n$. If $k' < 2n - k$ then $n - k > k' - n > 0$, so $(k - n) \mid (k' - n)$ doesn't hold. This case has been discussed already in Lemma 4. It is therefore sufficient to consider the case $k' \geq 2n - k$. Then we define a frame $\mathcal{F} = (W, R)$ as follows;

$$\begin{aligned} W &= \{w_i \mid 0 \leq i \leq 2n-k\}, \\ w_i R w_j &\Leftrightarrow |i-j| \leq 1. \end{aligned}$$

Since $2n - k > n$ by $n - k > 0$, we cannot have $w_0 R^n w_{2n-k}$ while $w_0 R^{k'} w_{2n-k}$ must hold, therefore $\mathcal{F} \notin \mathcal{PE}_{k'}^{m,n}$. On the other hand, if $xR^k y$ then $xR^n y$ for any $x, y \in W$, since $n > k$. Thus $\mathcal{F} \in \mathcal{PE}_k^{m,n}$. □

3. Concluding remarks

For non-negative integers m and n, we have shown when $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ holds. An interesting generalization of our results is what happens if we allow both m and n to change. More precisely, let $\mathbf{E}_k^{m,n}$ be the logic which is obtained from the smallest normal logic $\mathbf{K}$ by adding the axiom $\Diamond^k \phi \to \Box^m \Diamond^n \phi$, where $k, m, n \geq 0$. Then it is to see when $\mathbf{E}_k^{m,n} \supseteq \mathbf{E}_{k'}^{m,n}$ holds.

This paper presented a result in the case that m and n are fixed. The other cases are left unanswered, that is, inclusions between pseud-Euclidean logics in the cases that two of the numbers k and m are fixed, and k and n are fixed, respectively.

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