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ON THE LATTICE OF P-CONSEQUENCES

A b s t r a c t. This paper is devoted to investigation of the lattice properties of p-consequences. Our main goal is to compare the algebraic features of the lattices composed of all p-consequences and all consequence operations defined on the same propositional language.

1. Preliminary remarks concerning p-consequence operation

A concept of p-consequence is supposed to be a formal tool in a description of plausible reasoning [2]. This kind of reasoning is assumed to weaken the requirement for consequence operation to two conditions only: reflexivity and monotonicity. The third clause for consequence operation, i.e. idempotency expresses the fact that the conclusions of conclusions of a given set are conclusions of that set as well. The same fact can be described in terms of degree of certainty - conclusion is true at least in the same degree as premises are. So, we do not require that the condition of idempotency is valid for any plausible consequence. This approach is strictly related to

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this proposed by Ajdukiewicz [1]. It can be said that we try to capture his notion by the formal tools.

Definition 1.1. By a *p-consequence operation* for a sentential language $\mathcal{L}$ we mean any function $Z : \mathcal{P}(L) \rightarrow \mathcal{P}(L)$ that is subject to the conditions (for all $X \subseteq L, \alpha \in L$):

(i) $X \subseteq Z(X)$;

(ii) $Z(X) \subseteq Z(Y)$ whenever $X \subseteq Y$.

Moreover, if a p-consequence operation Z fulfils the condition

(iii) $Z(X) = \bigcup\{Z(X_f) : X_f \in Fin(L),\ X_f \subseteq X\}$, ($Fin(L)$ indicates the family of all finite sets (including the empty set) of formulas),

then Z will be called *finitary*.

We can additionally define the property of structurality for p-consequence Z: Z is *structural* iff $eZ(X) \subseteq Z(eX)$ for every $X \subseteq L$ and every endomorphism e of the language $\mathcal{L}$.

Obviously, any consequence operation is a p-consequence.

The most important way of representation of p-consequence (and the most intuitive) is that founded on the notion of *p-matrix* see [2]. p-matrix $\mathcal{M} = (\mathcal{A}, D_1, D_*)$ consists of an algebra $\mathcal{A} = (M, F_1, \ldots, F_n)$ similar to a propositional language and two distinguished sets $D_1 \subseteq D_* \subseteq M$. Every p-matrix defines a structural p-consequence $Z_{\mathcal{M}}$ in the following manner:

for every subset of formulas X and a formula α:

$\alpha \in Z_{\mathcal{M}}(X)$ iff $\overrightarrow{h}(X) \subseteq D_1$, then $h(\alpha) \in D_*$, for every homomorphism h of algebras $\mathcal{L}$ and $\mathcal{M}$.

When the distinguished sets are equal, then the p-matrix can be regarded as an ordinary matrix that defines a consequence operation (an opposite statement does not hold).

By distinguishing two sets in a p-matrix, we indicate which values are assumed to represent the true in a strong meaning (a smaller set D_1) and which set (D_* in our case) contains all values which are not the values of rejecting.

For the given formula α and a valuation h a metastatement "$h(\alpha) \in D_1$"

expresses the fact that under this valuation α is *absolutely* true. In the same way "$h(\alpha) \in D_*$" means that the formula is *possibly* true [1].

So, putting the definition of p-matrix and the above considerations together, we can say that a formula α is a p-consequence of X iff every interpretation which takes all formulas from X as absolutely true, it takes α as true in a weaker way (plausible). This statement reflects idea of Ajdukiewicz concerning plausible reasoning (see [1]).

Moreover - this way of representing of structural p-consequences is adequate in the meaning that the following holds:

Theorem 1.2. (see [2])*For every structural p-consequence Z, there exits a family of p-matrices $\mathcal{M}_t$ such that: $Z(X) = \bigcap_{t \in T} Z_t(X)$ for every set of formulas X. Where for $t \in T$, Z_t is a p-consequence determined by $\mathcal{M}_t$.*

The above theorem is in the fact a counterpart of the famous Lindenbaum lemma.

It is worth mentioning that p-matrix is a notion symmetrical to *q-matrix* (see [5],[6]). In our terminology q-matrix is an algebraic structure $\mathcal{M} = (\mathcal{A}, D_*, D_1)$ where $D_1 \subseteq D_*$ are the same like in a case of p-matrix. Each q-matrix defines *q-consequence* operation, i.e. such one for which the following clauses are valid:

(i) $N(X) \subseteq N(Y)$ for $X \subseteq Y$; (ii) $N(X \cup N(X)) \subseteq N(X)$.

Comparative studies concerning p- and q- consequence are contained in [7] and [2]. What is more - in a case of p-matrix, a set of premisses is valuated in a narrower set D_1 and a conclusion is valuated in a wider D_*. In a case of q-consequence the mentioned order is reversed - conclusion must be better than assumptions. Obviously there are more possibilities of such relations between the distinguished sets, i.e. when the set of antidesignated elements is not just a complementation of designated ones. These cases (together with q- and p- consequence) are considered in [8].

p-consequence can be easily described in a syntactic way. We think that this approach explains the best intuitions that are formalized by a notion

[1] A word *possibly* is assumed to bring some intuitions only. Probability theory is not "complete" interpretation of p-consequence.

of p-consequence. Let us remark a few important definitions from [2]:

Definition 1.3. Finite sequence $(a_1, \ldots, a_n), n \geq 1$, of ordered pairs from the Cartesian product $L \times \{*, 1\}$ will be called *p-inference for the language* $\mathcal{L}$. By a *p-rule of inference* for the language $\mathcal{L}$ we mean any nonempty set of p-inferences for $\mathcal{L}$.

For example, the sequence $(\langle p \to q, *\rangle, \langle p, 1\rangle, \langle q, *\rangle)$ is a p-inference while the set $\{(\langle(\alpha \to \beta, x_1\rangle, \langle \alpha, x_2\rangle, \langle \beta, *\rangle) : \alpha, \beta \in L, x_1, x_2 \in \{*, 1\}, x_1 = 1$ or $x_2 = 1\}$ is a p-rule of inference.

Definition 1.4. *p-proof of a formula* α *from a set* X *of formulas based on a set* $\mathcal{R}$ *of p-rules of inference* is a p-inference $(a_1, \ldots, a_k), k \geq 1$, for $\mathcal{L}$ such that

(i) $pr_1(a_k) = \alpha$;

(ii) for all $i = 1, 2, \ldots, k$, either $pr_1(a_i) \in X$ and $pr_2(a_i) = 1$ or there exists a p-rule $r \in \mathcal{R}$ and a p-inference $(b_1, \ldots, b_j) \in r$ such that $a_i = b_j$ and $\{b_1, \ldots, b_{j-1}\} \subseteq \{a_1, \ldots, a_{i-1}\}$. ($pr_1, pr_2$ are the first and the second projections on $L \times \{*, 1\}$, respectively).

Definition 1.5 A formula α is *p-derivable* from a set of formulas X by the p-rules from $\mathcal{R}$ ($X \Vdash_{\mathcal{R}} \alpha$ in symbols) iff there is a p-proof of α from X on the basis of $\mathcal{R}$. The relation $\Vdash_{\mathcal{R}}$ will be called a *p-derivability* relation determined by the set of p-rules $\mathcal{R}$.

For a given p-inference $(a_1, \ldots, a_k)$ one can say that the formulas appearing with an index 1 are these ones that are derived in a strong meaning - for example when they are the elements of a set of premisses X. When a formula α occurs in that sequence with index $*$, that is $\langle \alpha, *\rangle$ is an element of the sequence, than we say that α is at least *plausible.* It can be mentioned, that q-consequences being operations symmetrical to p-consequences can be syntactically described in a similar way (see [3]). However, the original approach and contained in [5] is based on ordinary notion of proof for consequence operation. The only difference is that q-proof does not include the condition that allows for adding to a proof formulas being the members of an initial set of premises.

For any p-inference $(a_1, \ldots, a_n)$ let us put for each $i \in \{1, \ldots, n\}$:

$A_*(i) = \{pr_1(a_l) : 1 \leq l \leq i \ \&\ pr_2(a_l) = *\}$ and $A_1(i) = \{pr_1(a_l) : 1 \leq l \leq i \ \&\ pr_2(a_l) = 1\}$. Assume that p-consequence Z on the language $\mathcal{L}$ is given. We are going to define the following set $\mathcal{R}(Z)$ of p-rules of inference:

$r \in \mathcal{R}(Z)$ iff for any $Y \subseteq L$ and p-inference $(a_1, \ldots, a_n) \in r$, the conditions: $A_*(n-1) \subseteq Z(Y), Z(Y \cup A_1(n-1)) = Z(Y)$ imply that

$(pr_2(a_n) = * \Rightarrow pr_1(a_n) \in Z(Y))$ & $(pr_2(a_n) = 1 \Rightarrow Z(Y, pr_1(a_n)) = Z(Y))$.

Now we are able to express a counterpart of well known theorem from theory of consequence:

Theorem 1.6. ([2]) *For any finitary p-consequence Z on the language $\mathcal{L}$, any $X \subseteq L$ and $\alpha \in L : \alpha \in Z(X)$ iff $X \Vdash_{\mathcal{R}(Z)} \alpha$.*

p-consequence operation might seems to be a very general notion. However, there exist many possibilities to limit the class of p consequences by putting some additional conditions. For example, one can consider strongly and weakly pseudoclosed p-consequences ([4]):

We shall say that p-consequence Z is *strongly* (*weakly*) *pseudoclosed* iff

$$\forall_{X,Y \subseteq L}(\forall_{\alpha \in Y}(Z(X, \alpha) = Z(X)) \Rightarrow Z(X \cup Y) = Z(X))$$

(for every finite $Y_f \subseteq L$
$\forall_{X \subseteq L}[\forall_{\alpha \in Y_f}(Z(X, \alpha) = Z(X)) \Rightarrow Z(X \cup Y_f) = Z(X)]$).

Although, the above notion is not a subject of this paper, let us try to decode these conditions. It is characteristic for p-consequence that the statement $Z(X, \alpha) = Z(X)$ is stronger than $\alpha \in Z(X)$. $Z(X, \alpha) = Z(X)$ can be seen as the other (but not equivalent to the considered so far) description of strong provability. So, pseudoclosureness of Z means that if all of formulas from Y do not extend a set of all conclusions of X, then the set Y does not add any new p-consequences, to the set $Z(X)$. Obviously every logical consequence has both of these properties.

Another class of p-consequences that has been taken under consideration are *deductive p–consequences* ([4]).

Definition 1.7. Any finitary p-consequence Z fulfilling:

(ded) $\forall_{X\subseteq L}\forall_{\beta\in L}\{Z(X,\beta) = Z(X) \Rightarrow \exists_{X_f\in Fin(X)}\forall_{Y\subseteq L}(Z(Y\cup X_f) = Z(Y) \Rightarrow Z(Y,\beta) = Z(Y))\}$.

is called *deductive*.

Lattice properties of pseudoclosed and deductive p-consequences have been described in [4], but due to limited size of this paper we have not decided to quote them here.

2. The poset $\langle \mathcal{Z}_{\mathcal{L}}, \leq\rangle$ and its the simplest properties.

Let us consider the set $\mathcal{Z}_{\mathcal{L}}$ of all p-consequences for a language $\mathcal{L}$. Let us define a binary relation $\leq\ \subseteq \mathcal{Z}_{\mathcal{L}} \times \mathcal{Z}_{\mathcal{L}}$ in the usual manner $Z_1 \leq Z_2$ iff $Z_1(X) \subseteq Z_2(X)$ for every $X \subseteq L$. It is obvious that $\leq$ is a partial order on $\mathcal{Z}_{\mathcal{L}}$.

Fact 2.1. *For any propositional language $\mathcal{L}$, the tuple $\langle \mathcal{Z}_{\mathcal{L}}, \leq\rangle$ is a complete, distributive, bounded lattice, with the greatest element Z_L, and the least one id (where id is the identity on $\mathcal{P}(L)$ and Z_L is constantly equal L). Moreover, for any $A \subseteq \mathcal{Z}_{\mathcal{L}}$ the operations of the greatest lower bound and the least upper bound $\bigwedge A$, $\bigvee A$ are defined on $\mathcal{P}(L)$ by the following clauses:*

$$(\bigwedge A)(X) \stackrel{\text{def}}{=} \bigcap_{Z\in A} Z(X) \text{ and } (\bigvee A)(X) \stackrel{\text{def}}{=} \bigcup_{Z\in A} Z(X).$$

Proof. Straightforward. □

Now we assume that a propositional language is fixed, so we can omit the lower index in $\mathcal{Z}_{\mathcal{L}}$. For a sake of convenience we assume the following notation: let $\mathcal{Z}_{str}, \mathcal{Z}_{fin}$ stand for the set of all structural and finitary p-consequences, respectively. Moreover for $Z_1, Z_2 \in \mathcal{Z}$, we put $Z_1 \wedge Z_2 := \bigwedge\{Z_1, Z_2\}$ and $Z_1 \vee Z_2 := \bigvee\{Z_1, Z_2\}$.

Fact 2.2.
a). *If $A \subseteq \mathcal{Z}_{str}$, then $\bigwedge A, \bigvee A \in \mathcal{Z}_{str}$, which means that $\langle \mathcal{Z}_{str}, \leq \rangle$ is a complete sublattice of $\langle \mathcal{Z}, \leq \rangle$.*
b). *If $Z_1, Z_2 \in \mathcal{Z}_{fin}$, then $Z_1 \wedge Z_2, Z_1 \vee Z_2 \in \mathcal{Z}_{fin}$, so $\langle \mathcal{Z}_{fin}, \leq \rangle$ is a sublattice of the lattice $\langle \mathcal{Z}, \leq \rangle$. Moreover, for any $A \subseteq \mathcal{Z}_{fin}$, $\bigvee A \in \mathcal{Z}_{fin}$.*

Proof. Easy. □

3. Comparison of $\mathcal{Z}$ and $\mathcal{C}$

Let $\mathcal{C}$ denote a set of all consequence operations on the language $\mathcal{L}$. Although $\langle \mathcal{C}, \leq \rangle$ is also a complete lattice with the same the greatest and the least elements, and $\mathcal{C} \subseteq \mathcal{Z}$, it is not a sublattice of $\langle \mathcal{Z}, \leq \rangle$. It follows from the fact, that $\langle \mathcal{C}, \wedge, \vee \rangle$ does not fulfil, contrary to $\langle \mathcal{Z}, \wedge, \vee \rangle$, law of distributivity. However, it is easy to see, that $\mathcal{C}$ is a meet-complete subsemilattice of $\mathcal{Z}$, i.e. for any $A \subseteq \mathcal{C}$: $\bigwedge A = \inf_{\mathcal{C}} A \in \mathcal{C}$, where $inf_{\mathcal{C}}$ is the greatest lower bound of A in the lattice of all consequences.

For partially ordered set $\langle A, \leq \rangle$ and $B, C \subseteq A$ we shall say that B is *dense in* C (w.r.t. $\leq$) iff for every $x, y \in C - B$: if $x < y$ then there exists $z \in B$ such that $x < z < y$.

Theorem 3.1. *The set $\mathcal{C}$ is not dense in $\mathcal{Z}$, i.e. there exist $Z_1, Z_2 \in \mathcal{Z} - \mathcal{C}$ such that $Z_1 < Z_2$ and there is no $C \in \mathcal{C}$ that fulfills $Z_1 < C < Z_2$.*

Proof. Let us choose from the set of propositional variables a countable subset and arrange it into a sequence $(p_i)_{i \in \mathbb{N}}$. Let us define $Z_1(X) := X \cup \{p_{i+1} : p_i \in X\}$, $Z_2(X) := X \cup \{p_{i+1} : p_i \in X\} \cup \{p_{i+2} : p_i \in X\}$. It is easy to see that Z_1, Z_2 are p-consequences fulfilling $Z_1 < Z_2$ and $Z_1, Z_2 \notin \mathcal{C}$. Assume that there exists a consequence C for which: $Z_1 < C < Z_2$. Hence: $p_2 \in Z_1(p_1) \subseteq C(p_1) \subseteq Z_2(p_1)$ and $p_3 \in Z_1(p_1, p_2) \subseteq C(p_1, p_2) = C(p_1) \subseteq Z_2(p_1)$. Similarly $Z_1(p_1, p_2, p_3) \subseteq C(p_1, p_2, p_3) = C(p_1, p_2) = C(p_1) \subseteq Z_2(p_1) = \{p_1, p_2, p_3\}$. Thus one can derive a contradiction $p_4 \in Z_1(p_1, p_2, p_3) \subseteq Z_2(p_1) = \{p_1, p_2, p_3\}$. □

Taking under consideration a structure $\langle \mathcal{Z} - \mathcal{C}, \leq \rangle$ as a substructure of

$\langle \mathcal{Z}, \leq \rangle$ we can make a statement that it does not form a sublattice of a lattice of all p-consequences. Moreover we are going to prove a stronger:

Theorem 3.2. *Every consequence C fulfilling the condition $|L - C(\emptyset)| > 1$ is the greatest lower bound of non-empty and finite set of p-consequences, which are not consequences.*

Proof. Let C be a consequence such that $|L - C(\emptyset)| > 1$. We divide this possibility on two cases:
i). $|L - C(\emptyset)| = 2$;
ii). $|L - C(\emptyset)| > 2$.

Ad i). Let $C(\emptyset) = L - \{\alpha, \beta\}$. Define two p-consequences:

$$Z_1(X) = \begin{cases} C(X) \cup \{\alpha\}, & \text{when } C(\emptyset) \subseteq X; \\ \\ C(X), & \text{in the other cases} \end{cases}$$

$$Z_2(X) = \begin{cases} C(X) \cup \{\beta\}, & \text{when } C(\emptyset) \subseteq X; \\ \\ C(X), & \text{in the other cases} \end{cases}$$

Reflexivity and monotonicity conditions (Definition 1.1.) trivially hold for Z_1 and Z_2. Moreover, $Z_1(Z_1(\emptyset)) = Z_1(C(\emptyset)) = Z_1(L - \{\alpha, \beta\}) = L - \{\beta\} \neq L - \{\alpha, \beta\} = Z_1(\emptyset)$. Thus $Z_1 \notin \mathcal{C}$. Similarly one can show $Z_2 \notin \mathcal{C}$.
Let $X \subseteq L$. By definition we obtain for any set $X \subseteq L: \ C(X) \subseteq Z_1(X) \cap Z_2(X) \subseteq (C(X) \cup \{\alpha\}) \cap (C(X) \cup \{\beta\}) = C(X) \cup (\{\alpha\} \cap \{\beta\}) = C(X)$. Finally, $C = Z_1 \wedge Z_2$.

Ad ii). Assume that there exist different $\alpha_1, \alpha_2, \alpha_3 \notin C(\emptyset)$. We will define the operations $Z_i : \mathcal{P}(L) \longrightarrow \mathcal{P}(L), \ i = 1, 2, 3$, in the following manner:

$$Z_i(X) = \begin{cases} C(X) \cup \{\alpha_i, \alpha_{i \oplus 1}\}, & \text{when } C(\emptyset) \cup \{\alpha_i\} \subseteq X \\ C(X) \cup \{\alpha_i\}, & \text{when } C(\emptyset) \cup \{\alpha_i\} \not\subseteq X \end{cases}$$

where $\oplus$ stands for a cyclic sum.

We are going to check, that for $i = 1, 2, 3, \ Z_i$ is a p-consequence that is not a consequence. Obviously $X \subseteq Z_i(X)$. When $X \subseteq Y$ and $C(\emptyset) \cup \{\alpha_i\} \subseteq$

X, then also $C(\emptyset)\cup\{\alpha_i\}\subseteq Y$, thus $Z_i(X)\subseteq Z_i(Y)$. (The other case, when $C(\emptyset)\cup\{\alpha_i\}\nsubseteq X$ is trivial). Moreover $Z_i(Z_i(\emptyset)) = Z_i(C(\emptyset)\cup\{\alpha_i\}) = C(C(\emptyset)\cup\{\alpha_i\})\cup\{\alpha_{i\oplus 1}\}\neq C(\emptyset)\cup\{\alpha_i\} = Z_i(\emptyset)$. So, Z_i, $i = 1,2,3$ are p-consequences and none of them is a consequence operation.

Now, we will show that $C = \bigwedge_{i=1}^{3} Z_i$. Let $X\subseteq L$. According to the fact that $\{\alpha_1,\alpha_2\}\cap\{\alpha_2,\alpha_3\}\cap\{\alpha_3,\alpha_1\} = \emptyset$ we obtain:

$$C(X)\subseteq(\bigwedge_{i=1}^{3} Z_i)(X)\subseteq\bigcap_{i=1}^{3}(Z_i(X)\cup\{\alpha_i,\alpha_{i\oplus 1}\}) =$$
$$(C(X)\cup\{\alpha_1,\alpha_2\})\cap(C(X)\cup\{\alpha_2,\alpha_3\})\cap(C(X)\cup\{\alpha_3,\alpha_1\}) =$$
$$C(X)\cup(\{\alpha_1,\alpha_2\}\cap\{\alpha_2,\alpha_3\}\cap\{\alpha_3,\alpha_1\}) = C(X). \qquad \Box$$

The assumption concerning the cardinality of the set $L - C(\emptyset)$ is relevant. For consider a consequence C for arbitrary but fixed language $\mathcal{L}$, for which $|L - C(\emptyset)|\leq 1$. First case when $L - C(\emptyset) = \emptyset$, concerns the inconsistent operation. This consequence is the greatest element in the lattice $\langle\mathcal{Z},\leq\rangle$, so it can not be g.l.b. of a non-empty set of p-consequences, containing an element different than C. Consider the case when $|L - (C(\emptyset))| = 1$ and define the consequence C:

$$C(X) = \begin{cases} L - \{\alpha\}, & \text{when } \alpha\notin X \\ L, & \text{when } \alpha\in X. \end{cases}$$

Now if Z is a p-consequence such that $C < Z$, then for some X fulfilling the clause $\alpha\notin X$, equation $Z(X) = L$ holds. Thus monotonicity of Z yields: $Z(L-\{\alpha\}) = L$, so $\bigcap_{C<Z\in\mathcal{Z}} Z(L-\{\alpha\}) = L$ while $C(L-\{\alpha\}) = L-\{\alpha\}$. Finally, consequence C, for which $|L - C(\emptyset)| = 1$ can not be g.l.b. of p-consequences different from C (that is p-consequences refuting idempotency condition).

The following is a counterpart of Theorem 3.1.:

Fact 3.3. *There exist $C_1, C_2\in\mathcal{C}_{fin}$ such that:*
(i) $C_1 < C_2$ (ii) *there is no $Z\in\mathcal{Z}$ such that $C_1 < Z < C_2$.*

Proof. Let us choose $p_0\in Var$ and put $C_1(\emptyset) = \emptyset$, $C_1(X) = X\cup\{p_0\}$, whenever $X\neq\emptyset$, and for every $X\subseteq L$, $C_2(X) = X\cup\{p_0\}$. Obviously $C_1, C_2 \in \mathcal{C}_{fin}$ and there is no p-consequence Z fulfilling the condition $C_1 < Z < C_2$. □

Obviously every p-consequence which is g.l.b. of some family of consequences is a consequence as well. The natural counterpart of Theorem 3.2. is

Theorem 3.4. *Every p-consequence is the least upper bound of some set of consequences. Moreover, every finitary p-consequence is l.u.b. of some finitary consequence operations.*

Proof. If p-consequence Z is a consequence, then $Z = \bigvee\{Z\}$. So, let $Z \in \mathcal{Z} - \mathcal{C}$. Consider the set $\{C \in \mathcal{C} : C < Z\}$. Of course $\bigvee\{C \in \mathcal{C} : C < Z\} \leq Z$. To prove the converse, that is $Z \leq \bigvee\{C \in \mathcal{C} : C < Z\}$, assume that $\alpha \in Z(X_0)$. Let C_0 be an operation defined by the conditions: $C_0(X) = X \cup \{\alpha\}$, when $X_0 \subseteq X$ and $C_0(X) = X$ otherwise. C_0 is a consequence, $\alpha \in C(X_0)$ and $C_0 < Z$ (since it could not be $C_0 = Z$, due to Z is not a consequence), thus $\alpha \in (\bigvee\{C \in \mathcal{C} : C < Z\})(X)$.

Let us go to the case when Z is a finitary p-consequence which is not consequence.

Consider the set $\{C \in \mathcal{C}_{fin} : C < Z\}$ for which also $\bigvee\{C \in \mathcal{C}_{fin} : C < Z\} \leq Z$ holds. For the converse, let for some $X_1 \subseteq L,\ \alpha \in L :\ \alpha \in Z(X_1)$. Thus $\alpha \in Z(X_0)$ for some finite $X_0 \subseteq X_1$, since Z is finitary. The operation C_0 defined as in the first part of the proof fulfills $\mathcal{C}_{fin} \ni C_0 < Z$, and $\alpha \in C_0(X_0) \subseteq C(X_1)$. Finally $\bigvee\{C \in \mathcal{C}_{fin} : C < Z\} = Z$. □

It is obvious (compare to Fact 3.1) that for any nonempty $\mathcal{C}' \subseteq \mathcal{C}$ the operation Z defined by the equation: $Z(X) := \bigcup\{C(X) : C \in \mathcal{C}'\}$ is a p-consequence. Thus by Theorem 3.4 - any p-consequence has exactly this form.

Similarly as in the theory of ordinary consequence operation we have the following:

Theorem 3.5. *For any p-consequence Z, operation $Z^* : \mathcal{P}(L) \longrightarrow \mathcal{P}(L)$ defined by the condition: $Z^*(X) = \bigcup\{Z(X_f) : X_f \in Fin(X)\}$ is the greatest finitary p-consequence Z' such that $Z' \leq Z$.*

Theorem 3.6. *For any p-consequence Z there exists a maximal element in the set of all finitary consequences C, such that $C \leq Z$.*

Proof. Without lost of generality we can assume that Z is not a consequence. Let $\emptyset \neq Ł$ be a chain in the poset $\langle\{C \in \mathcal{C}_{fin} : C < Z\}, \leq\rangle$. By Zorn lemma it is enough to show, that $\bigvee Ł \in \mathcal{C}_{fin}$. Obviously $\bigvee Ł \in \mathcal{Z}_{fin}$ (Fact 2.2.b). Assume that $\alpha \in (\bigvee Ł)((\bigvee Ł)(X))$. This implies $\alpha \in C_1((\bigvee Ł)(X))$ for some element C_1 of the chain Ł. Because C_1 is finitary, we have $\alpha \in C_1(C_2(X))$ for some $C_2 \in Ł$. Due to $C_1 \leq C_2$ or $C_2 \leq C_1$ we obtain $\alpha \in C_1(X)$ or $\alpha \in C_2(X)$. Finally – $\alpha \in (\bigvee Ł)(X)$. □

According to Theorem 3.5. we can say, that if Z is a consequence, then the greatest finitary p-consequence Z^* among those p-consequences Z' which validate $Z' \leq Z$, is a consequence. It is the only maximal element among all finitary consequences C such that $C \leq Z$.

4. Definability of p-consequence by its theories

Every operation of logical consequence is uniquely defined by the family of its theories, i.e. by the set $Th(C) = \{X \in \mathcal{P}(L) : X = C(X)\}$. In this paragraph we will show that this statement is not valid for p-consequence operation.

Fact 4.1. *For any propositional language $\mathcal{L} = (L, f_1, \ldots, f_n)$ there exists a family of p-consequences of power $2^{\mathfrak{c}} = |\mathcal{P}(L)^{\mathcal{P}(L)}|$: $\{Z_t\}_{t \in T}$ such that $\{X \in \mathcal{P}(L) : X = Z_t(X)\} = \{\emptyset, L\}$ for any $t \in T$.*

Proof. First of all we will proof our statement for a language without any connective (i.e. just consisting with countable set of propositional variables). Of course we can assume that $Var = \{p_{\bar{u}} : \bar{u} \in \mathbb{N}^*\}$, where $\mathbb{N}^*$ stands for the set of all finite sequences of natural numbers (including the empty set).

For any infinite set of natural numbers A let $\widehat{A}$ be a sequence of all elements from the set A ordered in a natural way.

For any set H of such $A's$ we define an operation Z_H as follows:

$$Z_H(X) = \begin{cases} \emptyset, & \text{whenever } X = \emptyset \\ X \cup \{p_{s(\bar{u})}\}_{p_{\bar{u}} \in X}\ , & \text{if } \exists_{A \in H} \forall_{p_{\bar{u}} \in X}\, \bar{u} \prec \widehat{A} \\ L\ (= Var), & \text{otherwise} \end{cases}$$

where $s(\langle u_1, \ldots, u_k \rangle) = \langle u_1 + 1, \ldots, u_k + 1 \rangle$ and $\prec$ is a prefix relation.

It is easy to check, that for any such H, operation Z_H is a p-consequence operation, such that its fixed point are exactly: $\emptyset$ and Var. It is enough to check that for different H, G, $Z_H \neq Z_G$ is true. Assume that $A \in H - G$, and $A = \{h_1, h_2, \ldots\}$, where $h_1 < h_2 < \ldots$. Then:

$$Z_H(\{p_{\langle h_1, h_2, \ldots h_i \rangle}\}_{i \in \mathbb{N}}) = Z_H(\{p_{\langle h_1, h_2, \ldots, h_i \rangle}\}_{i \in \mathbb{N}} \cup \{p_{\langle h_1 + 1, h_2 + 2, \ldots, h_i + 1 \rangle}\}_{i \in \mathbb{N}})$$
$$\neq L = Z_G(\{p_{\langle h_1, h_2, \ldots h_i \rangle}\}_{i \in \mathbb{N}})$$

We have proved the fact for a language without connectives. Now we will show that it holds in a general case. Let a propositional language $\mathcal{L} = (L, f_1, \ldots, f_n)$ be given. Then from the fact, that $|Var| = |L| = \aleph_0$, there exists a bijection $f : L \longrightarrow Var$. Now checking that operations $Z^f_H : \mathcal{P}(L) \longrightarrow \mathcal{P}(L)$, where: $\alpha \in Z^f_H(X)$ iff $f\alpha \in Z_H(\overrightarrow{f}(X))$ have required properties we finishes the proof. □

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