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There exist one-dimensional transitive cellular automata with non-empty set of strictly temporally periodic points

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Abstract. In a Cantor metric space $B^{\mathbb{Z}}$, we present a one-sided cellular automaton which positively answers the question
Does it exist a transitive cellular automaton $(B^{\mathbb{Z}}, F)$ *with non-empty set of strictly temporally periodic points?*
The question can be found in a current and recognized literature of the subject.

Keywords: one-dimensional cellular automata, transitive, strictly temporally periodic points.

1. Introduction

Dynamics of cellular automata have been investigated widely in mathematics, physics and theoretical computer science. In metric Cantor space of right infinite words $B^{\mathbb{N}}$ or bi-infinite words $B^{\mathbb{Z}}$ there are presented constructions of one-dimensional, transitive cellular automata [1, 4, 6] having or not some specified properties. The research on the dynamics of transitive cellular automata in $B^{\mathbb{N}}$ $(B^{\mathbb{Z}})$ is still carried out [1, 7]. In particular, two statements about transitive linear cellular automata and positively expansive automata $(B^{\mathbb{Z}}, F)$ are presented in [7]. In these statements occurs a notion of a strictly temporally periodic point which means a point in $B^{\mathbb{Z}}$, which is F-periodic and not σ-periodic. The set of all strictly temporally periodic points of a cellular automaton $(B^{\mathbb{Z}}, F)$ is denoted $STP(F)$. The authors are convinced that a positively expansive or linear and transitive cellular automaton $(B^{\mathbb{Z}}, F)$ has no strictly temporally periodic points.

A positively expansive cellular automaton $(B^{\mathbb{N}}(B^{\mathbb{Z}}), F)$ is transitive [4, 14]. Transitiviy is connected with the notion of chaos in the Devaney sense. It is true that a cellular automaton $(B^{\mathbb{N}}(B^{\mathbb{Z}}), F)$ is chaotic in the Devaney sense if and only if is transitive and its set of F-periodic points is dense [2]. The authors consider in [7] some arguments for the hypothesis that any transitive cellular automaton $(B^{\mathbb{Z}}, F)$ is chaotic in the Devaney sense if and only if it has no strictly temporally periodic points. They formulate a natural question
Question 3.7 – What is the largest class of sensitive CA where the set of strictly temporally periodic points is empty? Is this class the one of topologically transitive CA? Or, if a CA $(B^{\mathbb{Z}}, F)$ is transitive, can $STP(F)$ be non empty?

In this paper we consider a strictly temporally periodic point also as an element of $B^{\mathbb{N}}$. This allows us to point out an example of a cellular automaton in Cantor metric space of right-infinite words [10] and then bi-infinite, that is in $B^{\mathbb{N}}$ and $B^{\mathbb{Z}}$ which, in both cases, positively answers *Question 3.7* [7]. Our justification is based on [10] and [1].

2. Preliminaries

We denote by $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ the sets of non-negative integers, integers and real numbers, respectively. $\#Y$ stands for the cardinality of a set Y. A finite and non-empty set B is referred as an alphabet. A finite (non-empty) word w over B is a function $w : [0, k] \longrightarrow B$ defined on a discrete interval $[0, k]$, where $k \in \mathbb{N}$. The length of a word w, denoted by $|w|$, is equal to the cardinality of its domain. The set of all such defined words with concatenation of words " $\cdot$ " is a free semigroup $(B^+, \cdot)$. The set of all words in B^+ with the length equal to n is denoted B^n. Denoting λ the unit element of concatenation of words (empty word) we obtain a free monoid $(B^*, \cdot)$. By the definition $|\lambda| = 0$.

A right-infinite (bi-infinite) word is a function on $[0, \infty) = \mathbb{N}$ $((-\infty, \infty) = \mathbb{Z})$ with values in B. The set of all right-infinite (bi-infinite) words is denoted by $B^{\mathbb{N}}(B^{\mathbb{Z}})$. It is convenient to extend naturally concatenation on pairs of words in $B^* \times B^{\mathbb{N}}$ obtaining as a result an infinite word in $B^{\mathbb{N}}$. We will use also in the sequel words defined on finite discrete intervals of the form $I = [i, j]$, where $i \leq j$, $i, j \in \mathbb{Z}$. If $i = j$, then we denote such degenerated interval by $[i, i], [i]$ or $\{i\}$. For two discrete intervals I, J such that $J \subset I$ and for a word u defined on I we denote by u_J the restriction of u to J.

In the sequel we assume that $\#B \geq 2$.

Define a metric $d : B^{\mathbb{N}}(B^{\mathbb{Z}}) \times B^{\mathbb{N}}(B^{\mathbb{Z}}) \to \mathbb{R}$ putting for any $x, y \in B^{\mathbb{N}}(B^{\mathbb{Z}})$

$$d(x, y) = \begin{cases} 2^{-i} & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

where $i = min\{|j| \geq 0 : x(j) \neq y(j)\}$.

The resulted topological space $B^{\mathbb{N}}(B^{\mathbb{Z}})$ is a metric Cantor space [1, 14].

Let us fix $r \in \mathbb{N}$ and assume that there is given a mapping $F' : B^{r+1} \to B$. For any $x \in B^{\mathbb{N}}(B^{\mathbb{Z}})$, $i \in \mathbb{N}(\mathbb{Z})$ there exists $w \in B^{r+1}$ such that $w(j) = x(i+j)$ for any $j \in [0, r]$. We put in this case $F'(x_{[i,\, i+r]}) = F'(w)$. A mapping $F : B^{\mathbb{N}}(B^{\mathbb{Z}}) \to B^{\mathbb{N}}(B^{\mathbb{Z}})$ such that $F(x)(i) = F'(x_{[i,\, i+r]})$ for any $x \in B^{\mathbb{N}}(B^{\mathbb{Z}})$ and $i \in \mathbb{N}(\mathbb{Z})$ is referred to as a one-sided cellular automaton [1, 4, 8]. A mapping $F' : B^{r+1} \to B$ is called a local rule of F.

Any one-sided cellular automaton $(B^{\mathbb{N}}(B^{\mathbb{Z}}), F)$ is continuous. A point $x \in B^{\mathbb{N}}(B^{\mathbb{Z}})$ is F-periodic if and only if there exists $n \in \mathbb{N} \setminus \{0\}$ such that $F^n(x) = x$. If $F(x) = x$, then x is a fixed point of F. A one-sided surjective cellular automaton F is transitive if and only if for any open and non-empty sets $U, V \subset B^{\mathbb{N}}(B^{\mathbb{Z}})$ there exists $n \in \mathbb{N} \setminus \{0\}$ such that $U \cap F^{-n}(V) \neq \emptyset$ [4, 16].

If $\sigma(x)(i) = \sigma'(x_{[i,\, i+1]}) = x(i+1)$ for any $x \in B^{\mathbb{N}}(B^{\mathbb{Z}})$ and $i \in \mathbb{N}(\mathbb{Z})$, then a one-sided cellular automaton $(B^{\mathbb{N}}(B^{\mathbb{Z}}), \sigma)$ is referred to as one-sided (two-sided) full shift [12, 14].

A word $x \in B^{\mathbb{N}}(B^{\mathbb{Z}})$, which is F-periodic and not σ-periodic is called a strictly temporally periodic point [7]. A word $x \in B^{\mathbb{N}}(B^{\mathbb{Z}})$ which is F-periodic and σ-periodic is referred to as a jointly periodic point [1, 5]. A mapping $F : B^{\mathbb{N}}(B^{\mathbb{Z}}) \to B^{\mathbb{N}}(B^{\mathbb{Z}})$ can be considered as a symbolic dynamical system $(B^{\mathbb{N}}(B^{\mathbb{Z}}), F)$, abbreviated here as SDS [14].

3. Example of a cellular automaton F

The main goal of this section is to present a cellular automaton F with the announced properties.

We fix the following alphabets $B = \{0, 1, 2, 3\}$, and $E = \{0, 1\} \subset B$. Remind that for $a \in \mathbb{R}$ the floor of a, denoted $\lfloor a \rfloor$ is a maximal integer not greater then a. For $k \in \mathbb{Z}$, $n \in \mathbb{N} \setminus \{0\}$ $k \, mod \, n = k - n \left\lfloor \frac{k}{n} \right\rfloor$.

For any $a, b \in B$ a local rule $F' : B^2 \to B$ of the one-sided cellular automaton $(B^{\mathbb{N}}(B^{\mathbb{Z}}), F)$ is of the following form:
$F'(ab) = (a + \lfloor a/2 \rfloor [\lfloor b/2 \rfloor + 1]) \, mod \, 2 + 2 \left((\lfloor a/2 \rfloor + \lfloor b/2 \rfloor) \, mod \, 2\right).$

From [1, 10] follow presented below properties of the one-sided cellular automaton $(B^{\mathbb{N}}(B^{\mathbb{Z}}), F)$.

Proposition 1 *[10] Cellular automaton $(B^{\mathbb{N}}, F)$ is transitive.*

Proposition 2 *[1] If $(B^{\mathbb{N}}, F)$ is transitive, then $(B^{\mathbb{Z}}, F)$ is transitive also.*

Let us fix two infinite words $y \in B^{\mathbb{N}}$ and $z \in B^{\mathbb{Z}}$ such that
$y(0) = 1$, $y(i) = 0$ for any $i \in \mathbb{N} \setminus \{0\}$,
$z_{[0,\, \infty)} = y$, $z(i) = 0$ for any $i \in \mathbb{Z} \setminus \mathbb{N}$.

Lemma 1 *For any $k \in \mathbb{N} \setminus \{0\}$ it holds $F(y) = y$, $\sigma^k(y) \neq y$ and $F(z) = z$ and $\sigma^k(z) \neq z$.*

Proof. Observe that $F'(ab) = a$ for any $a, b \in E$. Thus $F(x) = x$ for any $x \in E^{\mathbb{N}}(E^{\mathbb{Z}})$ in particular $F(y) = y$ $(F(z) = z)$. For $k \in \mathbb{N} \setminus \{0\}$ assume that $\sigma^k(y) = u \in B^{\mathbb{N}}$. Hence $u(i) = 0$ for any $i \in \mathbb{N}$. Thus $\sigma^k(y) = u \neq y$ for any $k \in \mathbb{N} \setminus \{0\}$. If $\sigma^k(z) = v \in B^{\mathbb{Z}}$ then $v(-k) = 1$ i $v(i) = 0$ for any $i \in \mathbb{Z} \setminus \{-k\}$. Thus $\sigma^k(z) = v \neq z$ for any $k \in \mathbb{N} \setminus \{0\}$.

Propositions 1 and 2 and Lemma 1 imply the subsequent corollary.

Corollary 1 *The word $y \in B^{\mathbb{N}}$ is a strictly temporally periodic point of the transitive cellular automaton $(B^{\mathbb{N}}, F)$. The word $z \in B^{\mathbb{Z}}$ is a strictly temporally periodic point of the transitive cellular automaton $(B^{\mathbb{Z}}, F)$.*

It is possible to prove that the one-sided cellular automaton $(B^{\mathbb{Z}}, F)$ has a dense set of jointly periodic points [5], what means that it is chaotic in the Devaney sense [2].

4. Conclusion

There exist one-dimensional, transitive cellular automata with non-empty sets of strictly temporally periodic points which are chaotic in the Devaney sense.

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