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Existence and Multiplicity of Solutions for Noncoercive Neumann Problems with p-Laplacian*

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Abstract. We consider a nonlinear Neumann elliptic equation driven by the p-Laplacian and a Carathéodory perturbation. The energy functional of the problem need not be coercive. Using variational methods we prove an existence theorem and a multiplicity theorem, producing two nontrivial smooth solutions. Our formulation incorporates strongly resonant equations.

Keywords: Palais–Smale condition, noncoercive functional, second deformation theorem.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. We study the following nonlinear Neumann problem:

$$\begin{cases} -\Delta_p u(z) \;=\; f\big(z, u(z)\big) \quad \text{a.e. in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

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Here Δ_p denotes the p-Laplace differential operator, defined by

$$\Delta_p u(z) \ = \ \operatorname{div}\left(\|\nabla u(z)\|^{p-2}\nabla u(z)\right) \quad \forall u \in W^{1,p}(\Omega)$$

(with $1 < p < +\infty$). Also, $f(z,\zeta)$ is a Carathéodory function, i.e., for all $\zeta \in \mathbb{R}$, the function $z \longmapsto f(z,\zeta)$ is measurable and for almost all $z \in \Omega$, the function $\zeta \longmapsto f(z,\zeta)$ is continuous.

The aim of this work is to prove existence and multiplicity results for problem (1), when the energy functional of the problem is noncoercive. In fact, our hypotheses on the reaction f incorporate in our framework equations which are strongly resonant at infinity. Such problems are of special interest, since they exhibit a partial lack of compactness. Recently, there have been some existence and multiplicity results for Neumann problems driven by the p-Laplacian. We mention the works of Anello [1], Filippakis–Gasiński–Papageorgiou [3], Motreanu–Papageorgiou [6], O'Regan–Papageorgiou [7], Wu–Tan [8]. In Anello [1] and Wu–Tan [8], the key hypothesis is that $p > N$ (low dimension problems). This condition implies that $W^{1,p}(\Omega)$ is embedded compactly in $C(\overline{\Omega})$ (Sobolev embedding theorem) and this is their key mathematical tool. In Filippakis–Gasiński–Papageorgiou [3] and Motreanu–Papageorgiou [6], the potential function is nonsmooth (hemivariational inequality) and the energy function is coercive. Finally in O'Regan–Papageorgiou [7], the energy function is bounded below but need not be coercive. In fact, the potential function

$$F(t,\zeta) \ = \ \int_0^\zeta f(z,s)\,ds$$

is unbounded below as $\zeta \to \pm\infty$. The authors prove a multiplicity theorem using the notion of homological linking.

2. Mathematical background

Let X be a Banach space and let X^* be its topological dual. In what follows, by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X). Let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We say that φ satisfies the Palais–Smale condition at level $c \in \mathbb{R}$, if the following is true:

> Every sequence $\{x_n\}_{n\geqslant 1} \subseteq X$, such that
>
> $$\varphi(x_n) \longrightarrow c \quad \text{and} \quad \varphi'(x_n) \longrightarrow 0 \text{ in} X^*,$$
>
> admits a strongly convergent subsequence.

The following result is an easy consequence of the above definition (see Gasiński–Papageorgiou [4, p. 650]).

THEOREM 1. *If* $\varphi \in C^1(X)$ *is bounded below,* $c = \inf_X \varphi$ *and* φ *satisfies the Palais–Smale condition at level* c*, then there exists* $x_0 \in X$*, such that* $c = \varphi(x_0)$*, i.e.* x_0 *is a critical point of* φ*.*

For $\varphi \in C^1(X)$ and $c \in \mathbb{R}$, we introduce the following sets:

$$\begin{aligned} \varphi^c &= \{x \in X : \ \varphi(x) \leqslant c\}, \\ K_\varphi &= \{x \in X : \ \varphi'(x) = 0\}, \\ K_\varphi^c &= \{x \in K_\varphi : \ \varphi(x) = c\}. \end{aligned}$$

The next result is a basic tool in the minimax theorems of the critical point theory and it is known as the "second deformation theorem" (see Gasiński–Papageorgiou [4, p. 628]).

THEOREM 2. *If* $\varphi \in C^1(X)$*,* $a \in \mathbb{R}$*,* $a < b \leqslant +\infty$*,* φ *satisfies the Palais–Smale condition for every* $c \in [a, b)$*,* φ *has no critical values in* (a, b) *and* $\varphi^{-1}(\{a\})$ *contains at most a finite number of critical points of* φ*, then there exists a homotopy* $h\colon [0,1] \times (\varphi^b \setminus K_\varphi^b) \longrightarrow \varphi^b$*, such that*
(a) $h(1, \varphi^b \setminus K_\varphi^b) \subseteq \varphi^a$*;*
(b) $h(t, x) = x$ *for all* $t \in [0, 1]$*, all* $x \in \varphi^a$*;*
(c) $\varphi(h(t, x)) \leqslant \varphi(h(s, x))$ *for all* $t, s \in [0, 1]$*,* $s \leqslant t$*, all* $x \in \varphi^b \setminus K_\varphi^b$*.*

Let $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ be the nonlinear map, defined by

$$\langle A(u), v \rangle = \int_\Omega \|\nabla u\|^{p-2} (\nabla u, \nabla v)_{\mathbb{R}^N} \, dz \quad \forall u, v \in W^{1,p}(\Omega). \tag{2}$$

From Gasiński–Papageorgiou [4, p. 746], we have

PROPOSITION 3. *The nonlinear map* $A\colon W^{1,p}(\Omega) \longrightarrow W^{1,p}(\Omega)^*$ *defined by (2) is bounded, continuous, strictly monotone, hence maximal monotone too and of type* $(S)_+$*, i.e. if*

$$u_n \longrightarrow u \quad \text{weakly in } W^{1,p}(\Omega)$$

and

$$\limsup_{n \to +\infty} \langle A(u_n), \ u_n - u \rangle \leqslant 0,$$

then $u_n \longrightarrow u$ *in* $W^{1,p}(\Omega)$*.*

In what follows by $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$, i.e.

$$\|u\| = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}} \quad \forall u \in W^{1,p}(\Omega).$$

Also, if $g\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a measurable function (for example a Carathéodory function), then

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \quad \forall u \in W^{1,p}(\Omega).$$

Finally, by $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$.

3. Existence theorem

The existence theorem will be obtained for a more general version of problem than (1). Namely, let $h \in L^\infty(\Omega)$ be such that

$$\int_\Omega h(z)\,dz \;=\; 0.$$

We consider the following nonlinear Neumann problem:

$$\begin{cases} -\Delta_p u(z) \;=\; f\big(z, u(z)\big) + h(z) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3}$$

We work with the Sobolev space $W^{1,p}(\Omega)$ and consider the following direct sum decomposition of this space

$$W^{1,p}(\Omega) \;=\; \mathbb{R} \oplus V,$$

where

$$V \;=\; \Big\{u \in W^{1,p}(\Omega) : \int_\Omega u\,dz = 0\Big\}.$$

Hence, every $u \in W^{1,p}(\Omega)$ admits a unique decomposition

$$u \;=\; \overline{u} + \widehat{u}, \quad \text{with } \overline{u} \in \mathbb{R} \quad \text{and} \quad \widehat{u} \in V.$$

Recall that the elements of V satisfy

$$\|\widehat{u}\|_p \;\leqslant\; c_0(N,p)\|\nabla\widehat{u}\|_p \quad \forall \widehat{u} \in V, \tag{4}$$

for some $c_0(N,p) > 0$ (this is the so called Poincaré–Wirtinger inequality; see Gasiński–Papageorgiou [4, p. 841]). In particular, (4) implies that

$$\widehat{u} \;\longmapsto\; \|\nabla\widehat{u}\|_p$$

is an equivalent norm on V.

For $h \in L^\infty(\Omega)$ with

$$\int_\Omega h(z)\,dz \;=\; 0,$$

we consider the following auxiliary Neumann problem:

$$\begin{cases} -\Delta_p u(z) \;=\; h(z) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{5}$$

Let $\psi\colon V \longrightarrow \mathbb{R}$ be the C^1-functional, defined by

$$\psi(\widehat{u}) \;=\; \frac{1}{p}\|\nabla\widehat{u}\|_p^p - \int_\Omega h(z)\widehat{u}(z)\,dz \quad \forall \widehat{u} \in V.$$

PROPOSITION 4. *Problem (5) has a unique solution $\widehat{u}_0 \in V \cap C^1(\overline{\Omega})$, which is the unique minimizer of ψ.*

Proof. By virtue of the Poincaré–Wirtinger inequality (see (4)), we see that ψ is coercive. Also, using the Sobolev embedding theorem, we see easily that ψ is sequentially weakly lower semicontinuous. Hence, by the Weierstrass theorem, we can find $\widehat{u}_0 \in V$, such that

$$\psi(\widehat{u}_0) \;=\; \inf\big\{\psi(\widehat{u}):\; \widehat{u} \in V\big\},$$

so

$$\psi'(\widehat{u}_0) \;=\; 0 \quad \text{in } V^*$$

and thus

$$\big\langle A(\widehat{u}_0), v\big\rangle \;=\; \int_\Omega hv\,dz \quad \forall v \in V. \tag{6}$$

Let

$$v(z) \;=\; y(z) - \frac{1}{|\Omega|_N}\int_\Omega y\,dz \quad \text{with } y \in W^{1,p}(\Omega).$$

Then $v \in V$ and from (6), we have

$$\big\langle A(\widehat{u}_0), y\big\rangle \;=\; \int_\Omega hy\,dz$$

(since $\int_\Omega h\,dz = 0$), so

$$A(\widehat{u}_0) \;=\; h \quad \text{in } W^{1,p}(\Omega)^*$$

(since $y \in W^{1,p}(\Omega)$ is arbitrary) and thus

$$\begin{cases} -\Delta_p \widehat{u}_0(z) \;=\; h(z) \quad \text{a.e. in } \Omega, \\[2mm] \dfrac{\partial \widehat{u}_0}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Nonlinear regularity theory (see Lieberman [5]) implies that

$$\widehat{u}_0 \;\in\; V \cap C^1(\overline{\Omega}).$$

The uniqueness of $\widehat{u}_0$ follows from the strict monotonicity of A (see Proposition 3). □

Now let us introduce our hypotheses on the reaction f:

$\underline{H}$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$ and

(i) we have

$$\big|f(z,\zeta)\big| \;\leqslant\; a(z)+c|\zeta|^{r-1} \quad \text{for almost all } z\in\Omega,\ \text{all } \zeta\in\mathbb{R},$$

with $a \in L^{\infty}(\Omega)_+$, $c>0$, $p<r<p^*$, where

$$p^* \;=\; \begin{cases} \dfrac{Np}{N-p} & \text{if} \quad p<N,\\[2ex] +\infty & \text{if} \quad p\geqslant N; \end{cases}$$

(ii) if

$$F(z,\zeta) \;=\; \int\limits_0^{\zeta} f(z,s)\,ds,$$

then

$$F(z,\zeta) \;\leqslant\; \xi(z) \quad \text{for almost all } z\in\Omega,\ \text{all } \zeta\in\mathbb{R},$$

with $\xi \in L^1(\Omega)$;

(iii) there exists $c_0 \in \mathbb{R}\setminus\{0\}$, such that

$$\int\limits_{\Omega} F(z,c_0)\,dz \;>\; 0.$$

EXAMPLE 5. *The following function satisfies hypotheses H (for the sake of simplicity we drop the z-dependence):*

$$f(\zeta) \;=\; \begin{cases} |\zeta|^{p-2}\zeta & \textit{if} \quad |\zeta|\leqslant 1,\\[2ex] -\dfrac{\zeta}{|\zeta|^{p+2}}+\dfrac{4\zeta}{(1+\zeta^2)|\zeta|} & \textit{if} \quad |\zeta|>1. \end{cases}$$

In this case the potential function F is given by

$$F(\zeta) \;=\; \begin{cases} \dfrac{1}{p}|\zeta|^p & \textit{if} \quad |\zeta|\leqslant 1,\\[2ex] \dfrac{1}{p|\zeta|^p}+4\arctan|\zeta|-\pi & \textit{if} \quad |\zeta|>1. \end{cases}$$

Let $\varphi\colon W^{1,p}(\Omega)\longrightarrow\mathbb{R}$ be the energy functional for problem (3), given by

$$\varphi(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p-\int\limits_{\Omega} F\big(z,u(z)\big)\,dz-\int\limits_{\Omega} h(z)u(z)\,dz \quad \forall u\in W^{1,p}(\Omega).$$

Evidently $\varphi\in C^1\big(W^{1,p}(\Omega)\big)$. Recall that for every $u\in W^{1,p}(\Omega)$, we have in a unique way

$$u \;=\; \overline{u}+\widehat{u} \quad \text{with } \overline{u}\in\mathbb{R},\ \widehat{u}\in V.$$

So, we can write

$$\varphi(u) \;=\; \psi(\widehat{u}) - \int\limits_{\Omega} F\big(z, u(z)\big)\,dz \quad \forall u \in W^{1,p}(\Omega)$$

(recall that $\int\limits_{\Omega} h\,dz = 0$).

Hypotheses H incorporate in our framework, problems which are strongly resonant at infinity. It is well known that such problem exhibit a partial lack of compactness (see Bartolo–Benci–Fortunato [2]). This is reflected in the next proposition. In what follows $\widehat{u}_0 \in V \cap C^1(\overline{\Omega})$ is the unique solution of problem (5), established in Proposition 4. Also

$$\beta \;=\; \int\limits_{\Omega} \limsup_{|\zeta|\to+\infty} F(z,\zeta)\,dz.$$

By virtue of hypothesis $H(ii)$, $\beta \in [-\infty, +\infty)$.

PROPOSITION 6. *If hypotheses H hold and*

$$c \;<\; \psi(\widehat{u}_0) - \beta \;=\; \xi^* \;\in\; (-\infty, +\infty],$$

then φ satisfies the Palais–Smale condition at level c.

Proof. Let $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ be a sequence, such that

$$\varphi(u_n) \;\longrightarrow\; c \;<\; \xi^* \tag{7}$$

and

$$\varphi'(u_n) \;\longrightarrow\; 0 \quad \text{in } W^{1,p}(\Omega)^*. \tag{8}$$

Recall that

$$u_n \;=\; \overline{u}_n + \widehat{u}_n \quad \forall n \geqslant 1,$$

with $\overline{u}_n \in \mathbb{R}$, $\widehat{u}_n \in V$. On account of (7), we have

$$\varphi(u_n) \;\leqslant\; M_1 \quad \forall n \geqslant 1,$$

for some $M_1 > 0$, so

$$\begin{aligned} M_1 \;&\geqslant\; \frac{1}{p}\|\nabla \widehat{u}_n\|_p^p - \int\limits_{\Omega} F(z,u_n)\,dz - \int\limits_{\Omega} h\widehat{u}_n\,dz \\ &\geqslant\; \frac{1}{p}\|\nabla \widehat{u}_n\|_p^p - c_1\|\nabla \widehat{u}_n\|_p - c_2 \quad \forall n \geqslant 1, \end{aligned} \tag{9}$$

for some $c_1, c_2 > 0$. Here we have used hypothesis $H(ii)$ and the Poincaré–Wirtinger inequality (see (4)). Since $p > 1$, from (9), we infer that

$$\text{the sequence } \{\widehat{u}_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega) \text{ is bounded.} \tag{10}$$

So, by passing to a suitable subsequence if necessary, we may assume that

$$\widehat{u}_n \;\longrightarrow\; \widehat{u} \quad \text{weakly in } W^{1,p}(\Omega), \tag{11}$$

$$\widehat{u}_n \;\longrightarrow\; \widehat{u} \quad \text{in } L^p(\Omega), \tag{12}$$

$$\widehat{u}_n(z) \;\longrightarrow\; \widehat{u}(z) \quad \text{for almost all } z \in \Omega \tag{13}$$

and

$$\left|\widehat{u}_n(z)\right| \leqslant \widehat{\eta}(z) \quad \text{for almost all } z \in \Omega, \text{ all } n \geqslant 1,$$

with $\widehat{\eta} \in L^p(\Omega)$.

Claim. The sequence $\{u_n\}_{n\geqslant 1} \subseteq W^{1,p}(\Omega)$ is bounded.

Arguing by contradiction and passing to subsequence if necessary, we may suppose that

$$\|u_n\| \longrightarrow +\infty.$$

Since

$$\|u_n\| \leqslant \|\overline{u}_n\| + \|\widehat{u}_n\| \quad \forall n \geqslant 1,$$

from (10), it follows that $|\overline{u}_n| \longrightarrow +\infty$ (recall that $\{\overline{u}_n\}_{n\geqslant 1} \subseteq \mathbb{R}$). We have

$$\left|u_n(z)\right| \geqslant |\overline{u}_n| - \left|\widehat{u}_n(z)\right| \geqslant |\overline{u}_n| - \widehat{\eta}(z) \quad \text{for almost all } z \in \Omega, \text{ all } n \geqslant 1$$

(see (11)), so

$$\left|u_n(z)\right| \longrightarrow +\infty \quad \text{for almost all } z \in \Omega.$$

Since $\widehat{u}_0 \in V$ is the minimizer of ψ (see Proposition 4), we have

$$\varphi(u_n) = \psi(\widehat{u}_n) - \int_\Omega F\big(z, u_n(z)\big) \geqslant \psi(\widehat{u}_0) - \int_\Omega F\big(z, u_n(z)\big)\, dz.$$

From (7) and the Fatou lemma (see hypothesis $H(ii)$), we have

$$\xi^* > c \geqslant \psi(\widehat{u}_0) - \int_\Omega \limsup_{n\to+\infty} F(z, u_n)\, dz = \psi(\widehat{u}_0) - \beta = \xi^*,$$

a contradiction. This proves the Claim.

By virtue of the Claim, passing to a subsequence if necessary, we may assume that

$$u_n \longrightarrow u \quad \text{weakly in } W^{1,p}(\Omega), \tag{14}$$

$$u_n \longrightarrow u \quad \text{in } L^r(\Omega). \tag{15}$$

From (8), we have

$$\left|\left\langle \varphi'(u_n),\ y\right\rangle\right| \leqslant \varepsilon_n \|y\| \quad \forall y \in W^{1,p}(\Omega),$$

with $\varepsilon_n \searrow 0$, so

$$\left|\left\langle A(u_n),\ y\right\rangle - \int_\Omega f(z, u_n) y\, dz - \int_\Omega hy\, dz\right| \leqslant \varepsilon_n \|y\| \quad \forall n \geqslant 1.$$

We choose $y = u_n - u \in W^{1,p}(\Omega)$. Then

$$\left|\left\langle A(u_n),\ u_n - u\right\rangle - \int_\Omega f(z, u_n)(u_n - u)\, dz - \int_\Omega h(u_n - u)\, dz)\right|$$

$$\leq\ \varepsilon_n \|u_n - u\| \quad \forall n \geqslant 1. \tag{16}$$

From (14), we have

$$\int_\Omega f(z,u_n)(u_n-u)\,dz \ \longrightarrow\ 0 \quad \text{and} \quad \int_\Omega h(u_n-u)\,dz \ \longrightarrow\ 0.$$

Therefore, if in (16) we pass to the limit as $n \to +\infty$, then

$$\lim_{n\to+\infty} \langle A(u_n),\ u_n - u\rangle \ =\ 0,$$

so

$$u_n \ \longrightarrow\ u \quad \text{in } W^{1,p}(\Omega)$$

(see Proposition 3) and so φ satisfies the Palais–Smale condition at any level $c < \xi^*$. □

Using this proposition, we can have an existence theorem for problem (1).

THEOREM 7. *If hypotheses H hold and*

$$\beta \ <\ \int_\Omega F(z,\widehat{u}_0)\,dz,$$

then problem (3) admits a nontrivial solution $u^* \in C^1(\overline{\Omega})$.

Proof. From Proposition 4 and hypothesis $H(ii)$, we have

$$\varphi(u) \ =\ \psi(\widehat{u}) - \int_\Omega F(z,u)\,dz \ \geqslant\ \psi(\widehat{u}_0) - \|\xi\|_1 \quad \forall u \in W^{1,p}(\Omega),$$

so φ is bounded below.

Let

$$m_\varphi \ =\ \inf\big\{\varphi(u):\ u \in W^{1,p}(\Omega)\big\} \ >\ -\infty.$$

Then

$$-\infty \ <\ m_\varphi \ \leqslant\ \varphi(\widehat{u}_0) \ =\ \psi(\widehat{u}_0) - \int_\Omega F(z,\widehat{u}_0)\,dz \ <\ \psi(\widehat{u}_0) - \beta,$$

so φ satisfies the Palais–Smale condition at level m_φ (see Proposition 6).

Theorem 1 implies that we can find $u^* \in W^{1,p}(\Omega)$, such that

$$\varphi(u^*) \ =\ m_\varphi \ \leqslant\ \varphi(c_0) \ <\ 0 \ =\ \varphi(0)$$

(see hypothesis $H(iii)$), so

$$u^* \ \neq\ 0.$$

Also, we have

$$\varphi'(u^*) \ =\ 0,$$

so

$$A(u^*) \ =\ N_f(u^*) + h$$

and thus $u^* \in C^1(\overline{\Omega})$ (see (7)) is a nontrivial solution of (3). □

REMARK 8. *A careful inspection of the above proof, reveals that hypothesis $H(iii)$ is needed only if $h = 0$, to guarantee the nontriviality of u^*.*

4. Multiplicity theorem

In this section we prove a multiplicity theorem for problem (1) (i.e., now $h = 0$). For this purpose, we strengthen the hypotheses on f as follows:

$\underline{H}'$: $f\colon \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a Carathéodory function, such that $f(z,0) = 0$ for almost all $z \in \Omega$, hypotheses $H'(i) - (iii)$ are the same as $H(i) - (iii)$ and

(iv) we have

$$\beta \;=\; \int_\Omega \limsup_{|\zeta|\to+\infty} F(z,\zeta) \;<\; 0$$

and there exists $\vartheta \in L^\infty(\Omega)_+$, $\vartheta \neq 0$, such that

$$\vartheta(z) \;\leqslant\; \liminf_{\zeta\to 0} \frac{F(z,\zeta)}{|\zeta|^p} \quad \text{uniformly for almost all } z \in \Omega;$$

(v) we have

$$F(z,\zeta) \;\leqslant\; \frac{\widehat{\lambda}_1}{p}|\zeta|^p \quad \text{for almost all } z \in \Omega,\ \text{all } \zeta \in \mathbb{R},$$

with $\widehat{\lambda}_1 > 0$ being the first nonzero eigenvalue of the negative Neumann p-Laplacian.

EXAMPLE 9. *The following function satisfies hypotheses H' (as before, for the sake of simplicity, we drop the z-dependence):*

$$f(\zeta) \;=\; \begin{cases} \widehat{\lambda}_1|\zeta|^{p-2}\zeta & \text{if} \quad |\zeta| \leqslant 1,\\[2ex] (\widehat{\lambda}_1+1)\dfrac{\zeta}{|\zeta|^{p+2}} - |\zeta|^{r-2}\zeta & \text{if} \quad |\zeta| > 1, \end{cases}$$

where $p < r < p^*$. In this case the potential function F is given by

$$F(\zeta) \;=\; \begin{cases} \dfrac{\widehat{\lambda}_1}{p}|\zeta|^p & \text{if} \quad |\zeta| \leqslant 1,\\[2ex] \dfrac{\widehat{\lambda}_1+1}{p}\dfrac{1}{|\zeta|^p} - \dfrac{1}{r}|\zeta|^r - \dfrac{r-p}{rp} & \text{if} \quad |\zeta| > 1. \end{cases}$$

Now the energy functional $\widehat{\varphi}\colon W^{1,p}(\Omega) \longrightarrow \mathbb{R}$ is given by

$$\widehat{\varphi}(u) \;=\; \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega F\big(z,u(z)\big)\,dz \quad \forall u \in W^{1,p}(\Omega).$$

Evidently $\widehat{\varphi} \in C^1\big(W^{1,p}(\Omega)\big)$.

THEOREM 10. *If hypotheses H' hold, then problem (1) has at least two nontrivial smooth solutions $u^*, v^* \in C^1(\overline{\Omega})$.*

Proof. Since $h = 0$, we have

$$\widehat{u}_0 \;=\; 0.$$

Therefore

$$\int\limits_{\Omega} F\big(z, \widehat{u}_0(z)\big)\, dz \;=\; 0.$$

Since by hypothesis $H'(iv)$, $\beta < 0$, we can apply Theorem 7 and have one nontrivial smooth solution $u^* \in C^1(\overline{\Omega})$.

By virtue of hypothesis $H'(iv)$, for a given $\varepsilon > 0$ we can find $\delta = \delta(\varepsilon) > 0$, such that

$$F(z,\zeta) \;\geqslant\; \big(\vartheta(z) - \varepsilon\big)|\zeta|^p \quad \text{for almost all } z \in \Omega, \text{ all } |\zeta| \leqslant \delta.$$

If $\widehat{c} \in [-\delta, \delta]$, then

$$\varphi(\widehat{c}) \;=\; -\int\limits_{\Omega} F(z, \widehat{c})\, dz \;\leqslant\; |\widehat{c}|^p \Big(\varepsilon |\Omega|_N - \int\limits_{\Omega} \vartheta\, dz\Big).$$

Choosing $\varepsilon \in \Big(0,\ \frac{1}{|\Omega|_N} \int\limits_{\Omega} \vartheta\, dz\Big)$, we see that $\varphi(\widehat{c}) < 0$ and so

$$\max\big\{\varphi(v) :\ v \in \overline{B}_R \cap \mathbb{R}\big\} \;=\; \mu_R \;<\; 0 \quad \forall R \in \big(0,\ \delta|\Omega|_N^{\frac{1}{p}}\big), \tag{17}$$

where

$$\overline{B}_R \;=\; \big\{u \in W^{1,p}(\Omega) :\ \|u\| \leqslant R\big\}.$$

We consider the set

$$C(p) \;=\; \Big\{u \in W^{1,p}(\Omega) :\ \int\limits_{\Omega} \big|u(z)\big|^{p-2} u(z)\, dz \;=\; 0\Big\}.$$

Then for every $u \in C(p)$, we have

$$\varphi(u) \;\geqslant\; \frac{1}{p}\|\nabla u\|_p^p - \frac{\widehat{\lambda}_1}{p}\|u\|_p^p$$

(see $H'(v)$), so

$$\inf_{C(p)} \varphi \;=\; 0 \tag{18}$$

(see Gasiński–Papageorgiou [4]).

Let us fix $r \in \big(0,\ \delta|\Omega|_N^{\frac{1}{p}}\big)$. Let

$$\Gamma \;=\; \big\{\gamma \in C\big(\overline{B}_r \cap \mathbb{R},\ W^{1,p}(\Omega)\big) :\ \gamma|_{\partial \overline{B}_r \cap \mathbb{R}} = id|_{\partial \overline{B}_r \cap \mathbb{R}}\big\}$$

and define

$$\widehat{c}_r \;=\; \inf_{\gamma \in \Gamma} \max_{v \in \overline{B}_r \cap \mathbb{R}} \varphi\big(\gamma(v)\big). \tag{19}$$

Note that

$$\big(\partial\overline{B}_r \cap \mathbb{R}\big) \cap C(p) \;=\; \emptyset.$$

Also, let $\gamma \in \Gamma$ and define

$$\sigma(\xi) \;=\; \int\limits_{\Omega} \big|\gamma(\xi)\big|^{p-2}\gamma(\xi)\,dz \quad \forall \xi \in \overline{B}_r \cap \mathbb{R}.$$

Then

$$\partial B_r \cap \mathbb{R} \;=\; \big\{ \pm r_0 = \pm r|\Omega|_N^{\frac{1}{p}} \big\}$$

and so

$$\sigma(-r_0) \;<\; 0 \;<\; \sigma(r_0).$$

By virtue of the Bolzano theorem, we can find $\widehat{\xi} \in \overline{B}_r \cap \mathbb{R}$, such that

$$\sigma(\widehat{\xi}) \;=\; \int\limits_{\Omega} \big|\gamma(\widehat{\xi})\big|^{p-2}\gamma(\widehat{\xi})\,dz \;=\; 0,$$

so

$$\gamma(\widehat{\xi}) \;\in\; C(p),$$

thus

$$\gamma\big(\overline{B}_r \cap \mathbb{R}\big) \cap C(p) \;\neq\; \emptyset$$

and finally

$$c_r \;\geqslant\; 0 \tag{20}$$

(see (18) and (19)). Suppose that $\{0, u^*\}$ are the only critical points of φ. We set

$$a \;=\; \inf\varphi \;=\; \varphi(u^*) \;<\; 0 \quad \text{and} \quad b \;=\; \varphi(0) \;=\; 0.$$

By virtue of Proposition 6 and hypothesis $H'(iv)$, we see that φ satisfies the Palais–Smale condition for every level $c \in [a, b]$. Also,

$$\varphi^{-1}(\{a\}) \;=\; \{u^*\}.$$

Therefore, we can apply the second deformation lemma (see Theorem 2) and have a homotopy

$$\widehat{h}\colon [0,1] \times (\varphi^b \setminus K_\varphi^b) \;\longrightarrow\; \varphi^b,$$

such that

$$\widehat{h}\big(1, \varphi^b \setminus K_\varphi^b\big) \;\subseteq\; \varphi^a \;=\; \{u^*\} \tag{21}$$

and

$$\varphi\big(\widehat{h}(t,u)\big) \;\leqslant\; \varphi\big(\widehat{h}(s,u)\big) \quad \forall s,t \in [0,1],\; s \leqslant t,\; \text{all } u \in \varphi^b \setminus K_\varphi^b. \tag{22}$$

We consider the map

$$\gamma_0\colon \overline{B}_r \cap \mathbb{R} \;\longrightarrow\; W^{1,p}(\Omega),$$

defined by

$$\gamma_0(u) \;=\; \begin{cases} u^* & \text{if } \;\|u\| \leqslant \frac{r}{2}, \\[2ex] \widehat{h}\Big(\dfrac{2(r-\|u\|)}{r}, \dfrac{ru}{\|u\|}\Big) & \text{if } \;\|u\| > \frac{r}{2}. \end{cases} \tag{23}$$

If $u \in \mathbb{R}$ and $\|u\| = \frac{r}{2}$, then

$$\widehat{h}\Big(\frac{2(r-\|u\|)}{r}, \frac{ru}{\|u\|}\Big) \;=\; h(1,2u) \;=\; u^*$$

(see (17) and (21)). Hence, from (23), we see that γ_0 is continuous. Also, if $u \in \partial\overline{B}_r \cap \mathbb{R}$, then

$$\gamma_0(u) \;=\; \widehat{h}(0,u) \;=\; u$$

(see (23)). Therefore $\gamma_0 \in \Gamma$. From (17), (22) and (23), we have

$$\varphi\big(\gamma_0(u)\big) \;<\; 0 \quad \forall u \in \overline{B}_r \cap \mathbb{R},$$

so

$$\widehat{c}_r \;<\; 0 \tag{24}$$

(see (19)).

Comparing (20) and (24), we reach a contradiction. This means that we can find $v^* \in K_\varphi$, such that $v^* \notin \{0, u^*\}$. Then

$$A(v^*) \;=\; N_f(v^*),$$

so

$$\begin{cases} -\Delta_p v^*(z) \;=\; f\big(z, v^*(z)\big) \quad \text{a.e. in } \Omega, \\ \dfrac{\partial v^*}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

(see (7)). Nonlinear regularity theory (see Lieberman [5]) implies that $v^* \in C^1(\overline{\Omega})$. This is the desired second nontrivial smooth solution of (1). □

5. References

[1] Anello G.; *Existence of infinitely many weak solutions for a Neumann problem*, Nonlinear Anal. 57, 2004, pp. 199–209.

[2] Bartolo P., Benci V., Fortunato D.; *Abstract critical point theorems and applications to some nonlinear problems with "strong" resonance at infinity*, Nonlinear Anal. 7, 1983, pp. 981–1012.

[3] Filippakis M., Gasiński L., Papageorgiou N.S.; *Multiplicity result for nonlinear Neumann problems*, Canad. J. Math. 58, 2006, pp. 64–92.

[4] Gasiński L., Papageorgiou N.S.; Nonlinear Analysis, Chapman and Hall/CRC Press, Boca Raton, FL, 2006.

[5] Lieberman G.M.; *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. 12, 1988, pp. 1203–1219.

[6] Motreanu D., Papageorgiou N.S.; *Existence and multiplicity of solutions for Neumann problems*, J. Differential Equations 232, 2007, pp. 1–35.

[7] O'Regan D., Papageorgiou N.S.; *The existence of two nontrivial solutions via homological local linking for the non-coercive p-Laplacian Neumann problem*, Nonlinear Anal. 70, 2009, pp. 4386–4392.

[8] Wu X.-P., Tan K.K.; *On existence and multiplicity of solutions of Neumann boundary value problems for quasi-linear elliptic equations*, Nonlinear Anal. 65, 2006, pp. 1334–1347.