Rail Track and Rail Vehicle Intelligent Monitoring System

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Abstract

Structural Health Monitoring is becoming more and more important for today's railway. Higher requirements are set for safety and availability of both trains and tracks, that can be achieved using new type of SHM systems. Proposed SHM system is based on vibration measurements during the rail vehicle operation simultaneously with GPS position and velocity estimation. Algorithm for rail track health assessment is formulated as inverse identification problem of dynamic parameters of rail – vehicle system. Track irregularities of tracks are identified using this procedure. The case study of the SHM system operates on rail vehicle is presented in the paper.

Keywords: rail vehicle, track, monitoring, SHM

1. Introduction

Increasingly demanding market environment for railway transport, where cost efficiency must combine with availability, reliability and safety, new maintenance technologies are required. The knowledge about health of both infrastructure and vehicles is crucial to introduce more effective maintenance technology in this case. In the past, maintenance activities have in general most often been conducted when a fault has occurred to repair the system [1, 2]. With experience and increasing knowledge of technical systems, maintenance activities have evolved towards a more

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preventive approach based on time intervals as the knowledge about the degradation of the systems and components have increased [2, 3]. These maintenance time intervals are based on, for example, usage time, distance or the amount of operations the systems have been exposed to. In the railway industry the maintenance intervals are often traditionally based on time or mileage, and these intervals are often based on earlier experience or on the supplier's specification.

This method of maintenance can be further improve, if the variations in wear can be monitored.

Today there are many commercial products for condition monitoring on railway vehicles [4]. Many of these products are wayside monitoring systems and not directly mounted on the vehicles. One of the most popular system is based on Wayside Wheel/Rail Load detector [5] which helps to measure wheel/rail forces via strain gauge sensors on the rails in selected reverse curves. They also can measure the angle of attack of each axle with respect to the rail – a parameter that, in combination with measured vertical and lateral forces, provides information regarding the steering capability of a track through curves. Dynamic forces at the wheel-rail interface are capable of causing significant damage to track components and may also damage other rolling stock axle bearings [6]. Dynamic forces are generated by out-of-round wheels mainly. This phenomena can be a reason of crack of sleepers, damage the rail head, and cause failure of rail by either growth of rolling contact fatigue cracks, growth of detail fractures, or fatigue and fracture of in-track welds [7, 8]. In the extreme, fracture of the rail can result in derailment. It is assumed that derailment is a worst case scenario and that financial loss begins well before catastrophic failure of components or vehicle derailment. The wheel impact monitor is installed on railway infrastructure to avoid given above problems [9, 10]. The objective of installing a wheel impact monitor is to limit the damage that out-of round wheels cause. There are two widely used non-contact methods of quantifying the extent of damage on the wheel. Strain-based systems, which quantify the force applied to the rail between two sleepers by a direct relationship between the applied load and the deflection of the web or foot of the rail, can be estimated by a mathematical model of the deflection of the rail [11, 12]. The other non-contact method of measuring damage on the wheel is to place accelerometers on the rail [13, 12]. The extent of damage on the wheel is inferred from the output of the accelerometer. The height of the wheel flange, relative to the rail head can provide an indication of both the radial profile of the wheel and the amount of wear. Since the tip of the wheel flange is unlikely to be damaged or worn, this provides a quantification of the size of skid flat or out-of-roundness [14]. If standard limits for the out-of-roundness of wheels exist, this then provides a direct indication of the need to machine a wheel. The relative flange height could be measured by light, laser or ultrasound, or by a mechanism which presses a plate against the flange. Bearing failure, in service, can be catastrophic. A seized bearing can rotate on its journal and shear the axle, resulting in a train derailment [15, 16]. In the area of monitoring of railway tracks and vehicles, some patents can be found. One of them is the US Patent 6951132 –

Rail and train monitoring system and method in which measuring system is joint to track every 10-15 miles and acoustic signals is used for detection of damages of track and vehicle.

In SHM systems there are some different algorithms for damage detection and assessment, one of them is recently developed a wavelet based method [17], fuzzy – logic based detection algorithms [18], and artificial neural network based methods [19]. The new idea of monitoring is presented in [20, 21], there is description of the method based on inverse identification formulation which helps to detect damage of a wheel [22]. In the method the response of track on wheel excitation is measured. The knowledge of the track model is used to identify contact force time history. Based on identified load history the wheel roughness is determined. Application of the method requires installation on a track special measurement equipment. But, the method is not applicable for track roughness assessment. Proposed by authors SHM system is based on vehicle vibration measurements and can be used at any location on a track if vehicle moving through this zone. Locations of track irregularities are estimated using GPS receiver in combination with dedicated system for irregularities detection. The idea of the monitoring system is a subject of next section of the paper.

2. Problem Formulation

The idea of monitoring system is based on estimation of track One of the problems to be solved during the development of described diagnostics system is lateral rail irregularities identification based on the vibrations accelerations measured on the vehicle. This is an inverse problem defined in the following way [1]: model of the system is known as well as the response of the system. Excitation in form of the rail irregularities is to be identified. The graphical presentation of the inverse problem type can be found in the Figure 1.

This type of problem can be solved with use of various methods described in literature. Generally the excitation identification algorithms can be divided into three groups depending on the system model. This classification looks as follows [9]:
– deterministic methods e.g. [6], [7], [9], [10], [11], [13],
– statistic methods e.g. [2], [3],
– methods based on the artificial intelligence e.g. [3], [4], [12].

Methods based on regression analysis are the most often applied statistic methods of force identification [2]. The methods consist of identifying the regressive model parameters, which describes the relationship between the input force and the response of the system or process parameters. From the group of statistic methods for force reconstruction, it is worth mentioning the inverse structural filter algorithm presented in [5].

Methods based on artificial intelligence are used, when there is insufficient information about the objects dynamics to use its deterministic model. The second
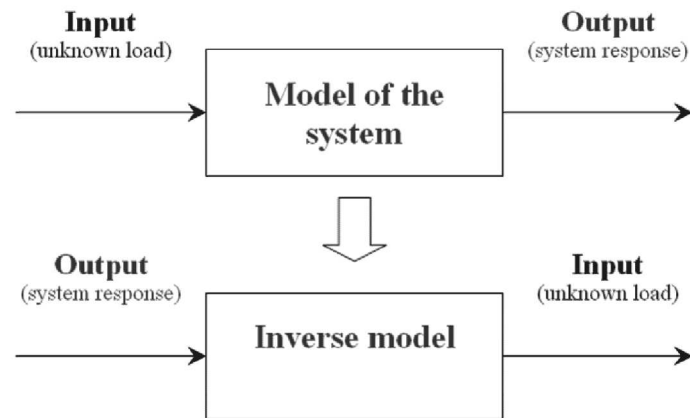
Fig. 1. Inverse problem presentation

case, when these methods are applied, is the problem of the deterministic model complexity – for example: it cannot be processed in real time. The advantage of artificial intelligence methods over statistic methods is their suitability for strong nonlinear cases. Also, when the object is too complex to be well described by the regressive model, artificial intelligence is applied. The artificial neural network [3], [4], plus fuzzy logic [4], are the methods most often used for force identification within this group of algorithms. Genetic algorithms are also applied for this purpose but often in combination with other algorithms [12].

The largest groups of algorithms are the methods based on deterministic dependencies. The problem can be solved here in the time frequency or amplitude domain. Within this method, developed in the time domain, one can distinguish the iteration methods and the single step methods, based on the mathematical dependencies. In this case, based on the system impulse response and the time histories of the responses, the time history of the excitation force is determined. One of the most often-used time domain methods is the method based on the deconvolution operation, where the properties of the Teoplitz matrix are utilized [13]. Another algorithm, which operates in the time domain, is the sum of weighted accelerations technique [6]. It identifies excitation forces by summing the acceleration responses with appropriate weights. In the time domain, one can identify the harmonic force by using the inversion of the parametric regressive model [7]. The time domain iterative methods deal with minimizing the quality coefficient which is defined as the difference between measured and estimated response of a system [9]. These methods are often preceded by regularization of the measured response data. Tikhonov is the most popular regularization algorithm [8].

In the frequency domain, a few techniques exist for excitation force identification. The first is the modal filter method [10], [11] which is based on the system modal model. The modal filter is a tool for extracting the modal coordinates of each

individual mode from the system outputs by mapping the response vector from the physical space to the modal space [10]. Application of the modal filter to force identification is carried out using four major steps [11]: transfer the outputs of the system from physical coordinates to the modal coordinates using a modal filter, determine the number of uncorrelated system inputs, locate these unknown inputs, calculate the amplitude of these inputs. The most popular method which operates in the frequency domain, is the frequency response functions (FRFs) matrix inversion technique. As a result of its application, the excitation force spectrum is identified [9].

In considered case it is necessary to find the kinematic excitation in form of lateral rail irregularities. This problem has to be solved in the function of path travelled by the vehicle, to be able to localize irregularities properly. However if the velocity of the train is known it is easy to find the relation between time of ride and the travelled distance. It is then suitable to apply one the algorithms, which identify the excitation in the time domain. As it was mentioned above it is a very complex problem due to the fact that it is non-linear and non-collocated. It would be convenient to use one of the method from the statistical group for example the inverse structural filter method [5]. The method was already used by the authors to solve the similar problem – rail-wheel contact force identification [14]. The method requires a lot of data to properly assess the structural filter parameters, and in the considered case there were no data available. For the same reason authors eliminated the method based on artificial intelligence, for example the artificial neural networks successfully applied earlier to load identification in nonlinear systems [4]. Taking into account all the difficulties and limitations mentioned above, authors decided to use the deterministic method based on the system model and objective function minimization, which was briefly presented above.

The method is one of the most often used iterative algorithms [13, 15]. It is especially dedicated to the time domain impulse loads identification. That means it is a perfect method for the considered case. The quality function is a measure of matching the measured response signal with the simulated one. The method bases on the minimization of the objective function:

$$\min J \rightarrow f \tag{1}$$

The objective function is defined as a difference between response of the system excited by the unknown force (irregularities) and response of the system model excited by the identified force (irregularities). The details will be describe later on.

The block diagram showing the identification procedure is presented in the Figure 2.

The tested rail vehicle was modeled in Adams multi-body software package. Below the main details considering the model used for simulation are gathered together. The model was composed of 7 bodies:
   – Car body;
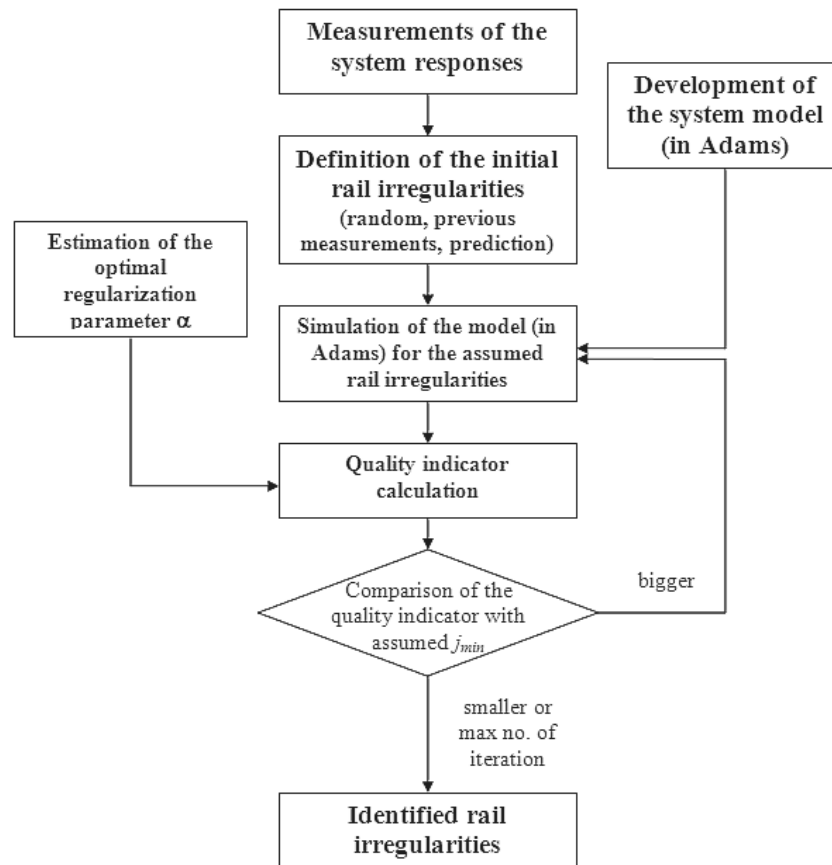   – Axels with wheels (wheel sets) (2 bodies);

Fig. 2. Procedure of lateral rail irregularities identification

– Axle boxes (4 bodies).

The following connections were modeled:

– Axel boxes with axels connected with use of the revolute joints;

– Car body with axel boxes connected with use of vertical springs and dampers (one spring and damper per one axel box).

The screen from the Adams containing the model is shown in the Figure 3.

During modeling it was necessary to keep the balance between accuracy of the model (it impacts the accuracy of the identification) and its simplicity (it decreases computation time).

3. Rail Vehicle and Track SHM System Architecture

Proposed monitoring system consists of three MEMS accelerometers, GPS module, microprocessor and memory module. The scheme of designed system is
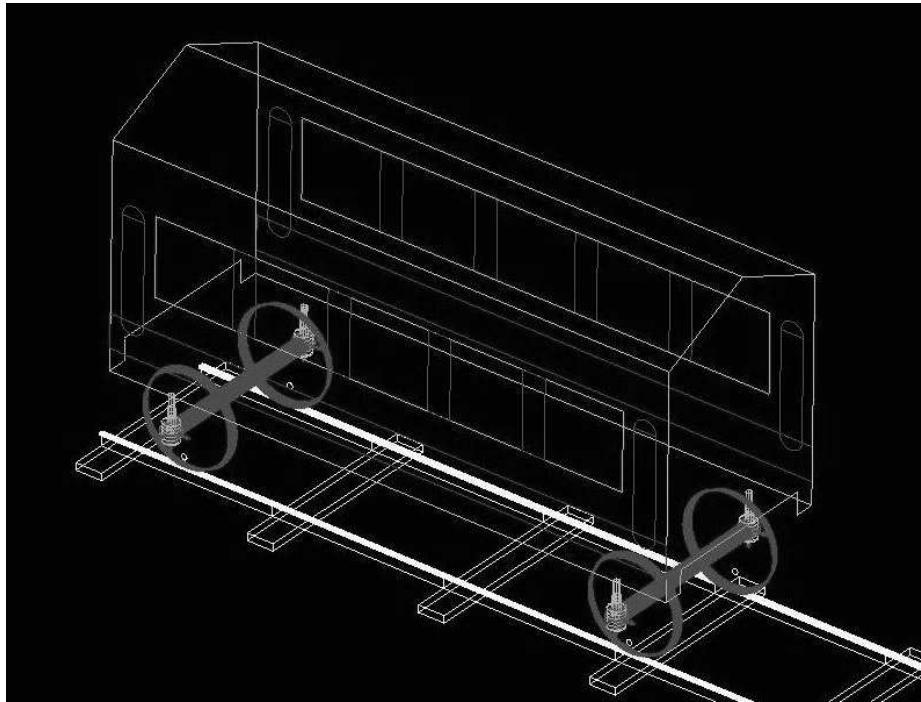
Fig. 3. Model of the object in Adams

shown in the Figure 4. The accelerometers are located on bearing box of wheelset, a bogie frame and vehicle under frame. Locations of the accelerometers are shown diagrammatically in the Figure 5. Location of accelerometers on bearing box base on the GPS receiver the location of the vehicle and its current speed can be estimated. The goal of the monitoring and diagnostic system is to detect, find location and assess dimensions of faults both tracks and vehicle.

The system should be installed on many vehicles running on the same track. As it is shown in the next section of this paper, based on simulation results, acceleration measured on bearing box is influenced by quality of the track. Solving of the inverse problem of track dynamics the geometry of the track can be assess. The location of track failures can be detected from GPS record. To avoid mistakes due to vehicle suspension failure or roughness of the wheel vibration increasing, several runs should be recorded and analyzed. If at each run at the same location on track, damage is detected by the system, with a high certainty it is caused by track roughness. To detect failures of rail vehicle driving and suspension systems the model based diagnostics can be employed. It is assumed that suspension has linear characteristic if its health is in a good condition, but if damage occurs the characteristics are changed and starts to have nonlinear form. Different type of nonlinearity can be find for different kind of suspension damages. The method of nonlinearity detection is implemented in real time system to have information about
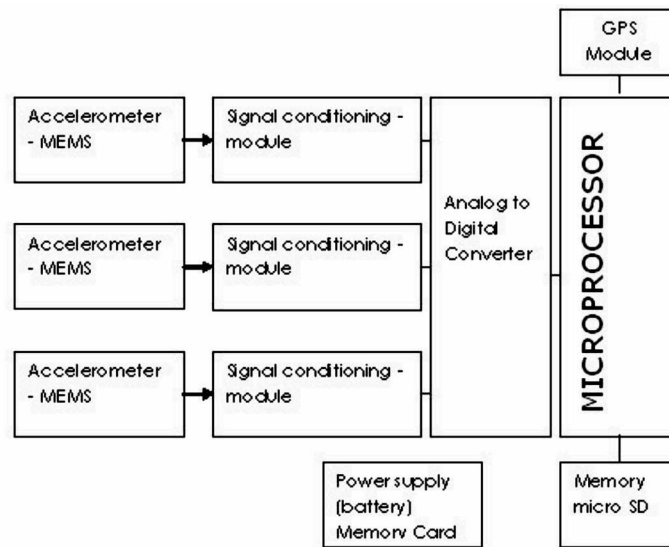
Fig. 4. Architecture of SHM system for rail vehicle and track monitoring

Fig. 5. Location of accelerometers on bearing box

possible failures on-line at the vehicle board but type of damages and its assessment can be obtained off line using software implemented on PC computer. To design of the detection and localization procedure the simulation study has been used.

4. Simulation Study

For the simulation study, a standard ERRI rail vehicle with two bogies and two axels each has been modeled (Figure 6). Each elastic element represents stiffness in three directions but dampers are defined according to given axis depends on location. For first level of suspension dampers are located In line with x and z directions but for second level of suspension spring are In line with Y and Z directions.

All mass and inertia elements are constant, but only springs and dampers parameters are treated as stochastic during simulation. Stochastic parameters have normal distribution with given mean values and variances. In this way uncertainties of structure due to technology problems are modeled. The following parameters of modeled vehicle are stochastic with normal distribution. Transverse stiffness of the primary suspension with mean value of k_{ps_xy} = 31400 N/mm and standard deviation of 5000 N/mm. Vertical stiffness of the primary suspension with mean value of k_{ps_z} = 1220 N/mm and standard deviation 210 N/mm. Transverse stiffness of the secondary suspension with mean value of k_{ss_xy} = 160 N/mm and standard deviation 20 N/mm.
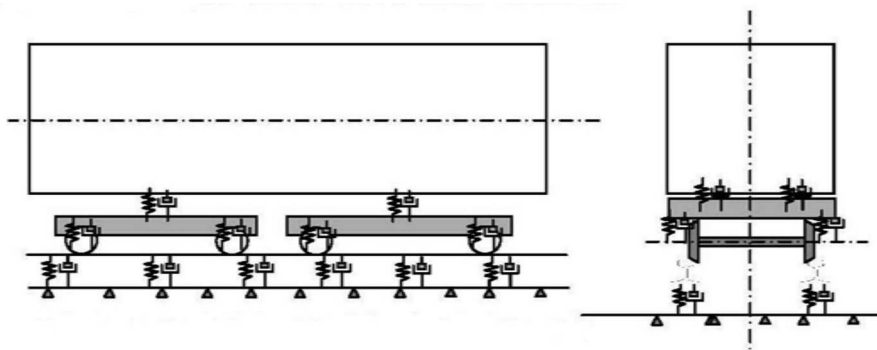

Fig. 6. Scheme of vehicle used for simulation study

Vertical stiffness of the secondary suspension with mean value of k_{ss_z} = 430 N/mm and standard deviation 65 N/mm. Vertical damping of the primary suspension with mean value of d_{pvd_z} = 1000 Ns/m and standard deviation 135 Ns/m. Transverse damping of the primary suspension with mean value of d_{pld_x} = 60000 Ns/m and standard deviation 9000 Ns/m. Vertical damping of the secondary suspension with mean value of d_{svd_z} = 6000 Ns/m and standard deviation 800 Ns/m. Transverse damping of the secondary suspension with mean value d_{sld_y} = 6000 Ns/m and standard deviation 800 Ns/m.

During the simulation, responses of the rail vehicle components in a form of acceleration at two points on bearing box and frame of the vehicle body. The vehicles drives with different speed starting from 40 [km/h] till 100 [km/h] on a rail with roughness in a form o regular waves with lengths of 2, 4, 6, 8 [m] simultaneously with equal amplitude of 5 [mm] with standard deviation of 1 [mm]. Monte Carlo Simulation scheme has been applied to find probability density function of acceleration at given point located on vehicle structure. The results are shown in Figure 7.

As it can be observed from obtained results for higher driving speed dispersion of obtained results is bigger what makes detection of rail truck quality more difficult. The dispersion of acceleration measured at bearing box is the smallest one and this location of accelerometers should be used for detection of track damages. The second

task for designed monitoring system is detection of faults of vehicle suspension primary and secondary one. In order to assess possibility of fault detection based on acceleration measurements sensitivity of acceleration.

The accelerations measured on bearing box are sensitive to change of suspension parameters and can be applied for damage detection of vehicle (see Figure 8).
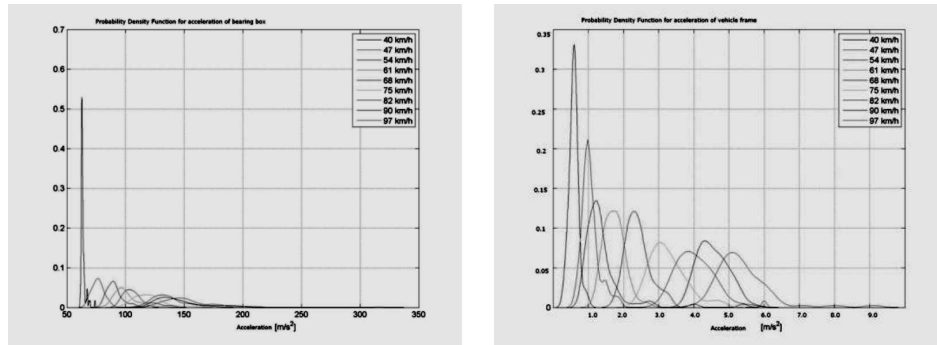

Fig. 7. Results of simulation of acceleration of vehicle a) bearing box, b) under frame of vehicle, moving with different speed on the truck with roughness mean value amplitude 5 [mm] and standard deviation of 1 [mm]
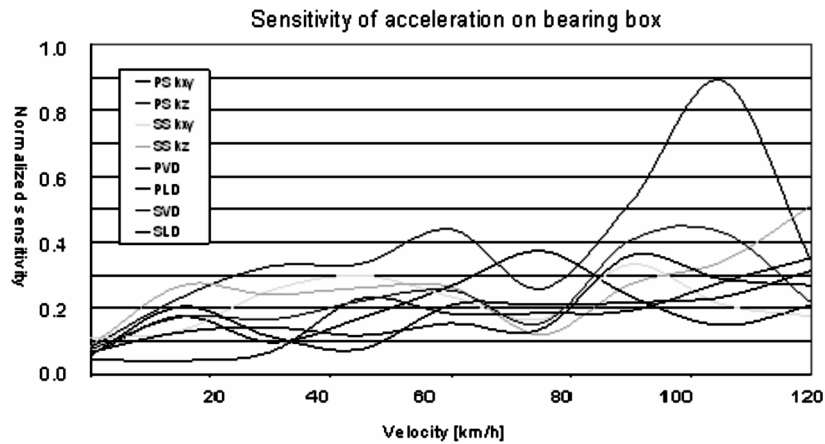

Fig. 8. Sensitivity of acceleration of bearing box on suspension parameters changes

5. Procedure of Rail Track Roughness Identification

Formulated track roughness identification is based on solution of an inverse identification problem, which is defined as follows [23, 24]; model of the system

is known as well as the response of the system, but excitation in form of the rail irregularities is to be identified.

Such defined problem is not easy to solve because a rail vehicle – rail track system is non-linear and non-collocated. The problem has to be solved in the time domain to find track irregularities. To its solution the method of quality function minimization was proposed [25]. The quality function is a measure of matching the measured response signal with the simulated one. The method bases on the minimization of the objective function: min J → f.

The objective function is defined as a difference between response of the system excited by the unknown track irregularities and response of the system model excited by the identified track irregularities:

$$J(x, f_j) = \sum_{j=1}^{N} (q_j - y_j) D(q_j - y_j)^T \tag{2}$$

where: c – initial conditions of the motion f_j, q_j, y_j – vectors of load (rail irregularities) and of the response calculated and measured in the time sample j, D – weight matrix.

Such a formulation of the problem is not sufficient because a mathematical solution that will minimize J will usually end up with the model exactly matching the data. The situation that is to be avoided. This is where the regularization method enters. By adding a term to the objective function one can control the amount of smoothness that occurs in the solution by varying the parameter a. This method is referred to as Tikhonov's method [26]. With the regularization term the full formula of the objective function presents as follows [27]:

$$J(c, f_j) = \sum_{j=1}^{N} (q_j - y_j) D(q_j - y_j)^T + \alpha f_j^T f_j \tag{3}$$

where: α – smoothing parameter.

Authors decided to use L – curve method [27, 28] for α parameter selection as the one which requires less computations. To solve the optimization problem the dynamic programming was used [29]. Entire procedure was programmed in MAT-LAB. The model of the vehicle was build in MD ADAMS. Formulated identification procedure was shown in the Figure 2.

6. Verification of the Method Using Simulation

Firstly the presented approach was tested on simulation data. The rail irregularities identification was predated by the procedure parameters assessment. The minimal value of the objective function was set to $j_{min} = 0.1$. The maximum number of iteration was set to 1000. The regularization parameter was estimated with use of

the L-curve method its value amounted 0.1E-03. Results of rail irregularities identification are presented in the Figure 9 – comparison of the acceleration simulated initially (measured) and obtained as a result of the iterative procedure, and in the Figure 10 – comparison of rail irregularities – assumed and identified.
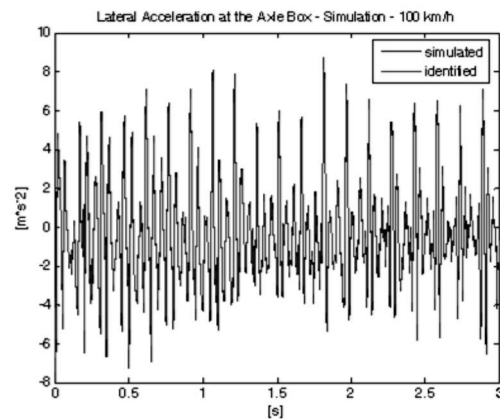

Fig. 9. Comparison of the acceleration simulated initially (measured) and obtained as a result of the iterative procedure
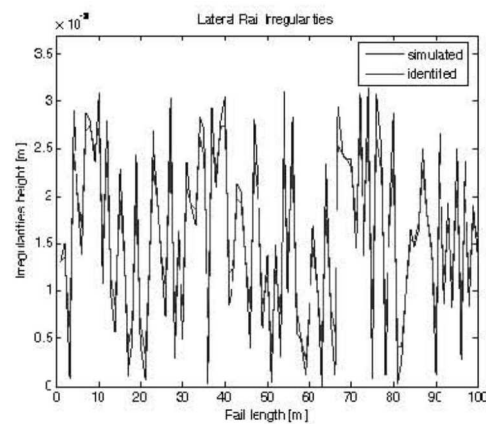

Fig. 10. Comparison of rail irregularities – assumed and identified

As it can be seen in the Figures 9 and 10 the results of simulation verification were quite promising. As a comparison criterion the correlation coefficient and the RMS of the signals were established. The correlation coefficient between rail irregularities assumed and identified amounted 0.9957. The error between RMS calculated for both signals was smaller than 17%.

7. Application of the Procedure to the Measured Data

The presented procedure was also applied to the measured data. The experiment was performed on the route between Krakow and Wadowice. The testing sector of the route is presented in the Figure 11. The lateral vibrations accelerations were measured on the axel boxes of the middle (B) boogie of the vehicle. The measurements were taken for different velocities (20-85 km/h) and different drive profiles (acceleration, braking, ride in curves). The rail irregularities were not measured.

That is why performed rail irregularities identification could have been judged only from the accelerations comparison. The calculations were done for different velocities of the vehicle-bus. In the Figure 12 the comparison between vibrations accelerations measured and simulated in Adams is placed.

In all runs one can notice that the procedure follows the data in general but the higher frequency component are not reconstructed well. It is due to the fact that in the model only lower modes are included. In the Figure 13 example of identified lateral rail irregularities is presented.

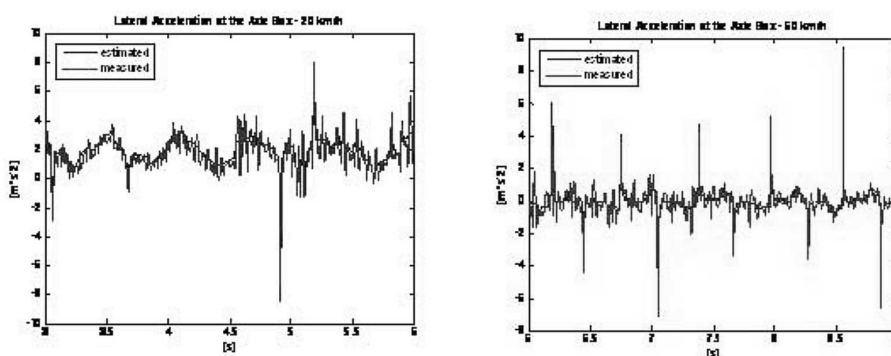Fig. 11. Measured track (photo by Google Maps) and rail vehicles used for test

Fig. 12. Comparison between vibrations accelerations measured and simulated in Adams for different velocities of the vehicle
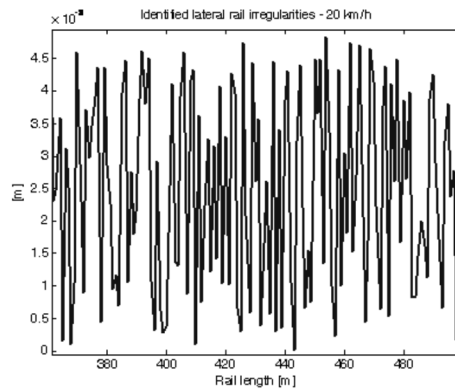
Fig. 13. Rail irregularities identified for 20 km/h – no measurement verification Both the character of irregularities and their amplitude have reasonable value

8. Conclusions and Final Remarks

The first attempt of lateral rail irregularities identification has been presented. Simulation verification gave relatively good results – correlation coefficient between simulated end identified runs amounted 0.99. The error between RMS calculated for both signals was lower than 17%. The identification performed on the real measurement data showed that the best results one can achieve for small velocities. However this conclusion is drawn only from the vibrations accelerations comparison – there was no rail irregularities measurements performed to compare. The method can be applied as SHM method for track condition monitoring and is helpful to rail track maintenance planning. For the further work we will test different optimization and regularization algorithms. The real time implementation of the procedure is planned. The experiment with measured rail irregularities is also planned to confirm efficiency of the method for the physical data.

Acknowledgments

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