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# Positivity and asymptotic stability of descriptor linear systems with regular pencils

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The positivity and asymptotic stability of the descriptor linear continuous-time and discrete-time systems with regular pencils are addressed. Necessary and sufficient conditions for the positivity and asymptotic stability of the systems are established using the Drazin inverse matrix approach. Effectiveness of the conditions are demonstrated on numerical examples.

**Key words:** descriptor, Drazin inverse, linear, system, positivity, asymptotic stability

## 1. Introduction

Descriptor (singular) linear systems have been considered in many papers and books [6-8, 10, 12, ,17, 22, 23]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [6-8, 13] and the realization problem for singular positive continuous-time systems with delays in [18]. The computation of Kronecker's canonical form of singular pencil has been analyzed in [22]. The positive linear systems with different fractional orders have been addressed in [16, 19]. Selected problems in theory of fractional linear systems has been given in monograph [19].

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in [5, 9, 15]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc. Descriptor standard positive linear systems by the use of Drazin inverse has been addressed in [1-4, 13]. The checking of the positivity of descriptor linear systems is addressed in [11, 17]. The stability of positive descriptor systems has been investigated in [20, 23].

In this paper the positivity and asymptotic stability of the descriptor linear systems is investigated. Necessary and sufficient conditions for the positivity and asymptotic stability of the continuous-time and discrete-time linear systems are established.

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The paper is organized as follows. In section 2 some preliminaries concerning standard positive linear systems are recalled. The positivity of autonomous descriptor systems is addressed in section 3 and the asymptotic stability of the system in section 4. An extension of the considerations for nonautonomous descriptor linear systems is presented in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $Z_+$ – the set of nonnegative integers, $\Re_+^{n\times m}$ – the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, $M_n$ – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), $I_n$ – the $n\times n$ identity matrix.

## 2. Preliminaries

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{1}$$

where $x(t)\in\Re^n$, $u(t)\in\Re^m$ are the state and input vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$.

**Definition 1** [9,15] *The system* (1) *is called (internally) positive if* $x(t)\in\Re_+^n$, $t\geqslant 0$ *for any initial conditions* $x(0)=x_0\in\Re_+^n$ *and all inputs* $u(t)\in\Re_+^m$, $t\geqslant 0$.

**Theorem 1** [9,15] *The system* (1) *is positive if and only if*

$$A\in M_n, \quad B\in\Re_+^{n\times m}. \tag{2}$$

**Definition 2** [9,15] *The positive system* (1) *is called asymptotically stable if for* $u(t)=0$, $t\geqslant 0$

$$\lim_{t\to\infty} x(t) = 0 \quad \text{for all} \quad x_0\in\Re_+^n. \tag{3}$$

**Theorem 2** [9,15] *The positive system* (1) *is asymptotically stable if and only if all coefficients of the polynomial*

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{4}$$

*are positive, i.e.* $a_i > 0$ *for* $i = 0,1,\ldots,n-1$.

Now let us consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \quad i\in Z_+ \tag{5}$$

where $x_i\in\Re^n$, $u_i\in\Re^m$ are the state and input vectors and $A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$.

**Definition 3** [9,15] *The system* (5) *is called (internally) positive if* $x_i \in \Re^n{}_+$, $i \in Z_+$ *for any initial conditions* $x_0 \in \Re^n$ *and all inputs* $u_i \in \Re^m$, $i \in Z_+$.

**Theorem 3** [9,15] *The system* (5) *is positive if and only if*

$$A \in \Re_+^{n\times m}, \quad B \in \Re_+^{n\times m}. \tag{6}$$

**Definition 4** [9,15] *The positive system* (5) *is called asymptotically stable if for* $u_i = 0$, $i \in Z_+$

$$\lim_{i\to\infty} x_i = 0 \;\; \textit{for all} \;\; x_0 \in \Re_+^n. \tag{7}$$

**Theorem 4** [9,15] *The system positive* (5) *is asymptotically stable if and only if all coefficients of the polynomial*

$$\det[I_n(z+1)-A] = z^n + \bar{a}_{n-1}z^{n-1} + \ldots + \bar{a}_1 z + \bar{a}_0 \tag{8}$$

*are positive, i.e.* $\bar{a}_i > 0$ *for* $i = 0,1,\ldots,n-1$.

## 3. Positivity of descriptor linear systems

### 3.1. Continuous-time system

Consider the descriptor autonomous continuous-time linear system

$$E\dot{x}(t) = Ax(t), \quad x(0) = x_0 \tag{9}$$

where $x(t) \in \Re^n$ is the state vectors and $E, A \in \Re^{n\times n}$.

It is assumed that rank $E < n$ and the pencil $Es - A$ is regular, i.e

$$\det[Es - A] \neq 0 \;\; \text{for some} \;\; s \in C \;\; \text{(the field of complex number)}. \tag{10}$$

Choosing $s = c$ such that $\det[Ec - A] \neq 0$ and pre-multiplying (9) by the matrix $[Ec - A]^{-1}$ we obtain

$$\hat{E}\dot{x}(t) = \hat{A}x(t) \tag{11}$$

where

$$\hat{E} = [Ec - A]^{-1}E, \quad \hat{A} = [Ec - A]^{-1}A. \tag{12}$$

It is easy to check that [13]

$$\hat{E}\hat{A} = \hat{A}\hat{E} \tag{13}$$

since $\hat{E}c - \hat{A} = [Ec - A]^{-1}[Ec - A] = I_n$ and $\hat{A} = \hat{E}c - I_n$.

**Definition 5** [4,13] *A matrix $A^D$ is called the Drazin inverse of $A \in \mathbb{C}^{n\times n}$ if it satisfies the conditions*

$$AA^D = A^D A, \tag{14a}$$

$$A^D AA^D = A^D, \tag{14b}$$

$$A^D a^{q+1} = A^q \tag{14c}$$

*where $q$ is the index of $A$, i.e. the smallest nonnegative integer $q$ such that*

$$\text{rank} A^q = \text{rank} A^{q+1}. \tag{15}$$

**Lemma 1** [13] *If* (13) *holds then*

$$\hat{E}^D \hat{A} = \hat{A}\hat{E}^D, \tag{16}$$

$$\hat{E}\hat{A}^D = \hat{A}^D \hat{E}, \tag{17}$$

$$\hat{E}^D \hat{A}^D = \hat{A}^D \hat{E}^D. \tag{18}$$

**Theorem 5** [4,13] *The solution to the equation* (11) *has the form*

$$x(t) = e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v \tag{19}$$

*where $v \in \Re^n$ is arbitrary.*

Note that (19) is the solution of the differential equation

$$\dot{x}(t) = \bar{A}x(t), \quad x(0) \in \text{im}(\bar{F}) \tag{20a}$$

where

$$\bar{A} = \hat{E}^D \hat{A}, \quad \bar{F} = \hat{E}^D \hat{E}, \quad x(t) = e^{\bar{A}t} \bar{F} v \tag{20b}$$

and $\text{im}(\bar{F})$ represents the image of $\bar{F}$. Therefore, the descriptor system described by (11) is equivalent to the standard system descried by (20).

**Theorem 6** *For the standard system* (20) *the following holds*

$$\bar{A}\bar{F} = \bar{F}\bar{A} \tag{21}$$

*and*

$$x(t) = \bar{F}x(t), \quad t \geqslant 0. \tag{22}$$

**Proof** Using (20b), Definition 5 and Lemma 1 we obtain

$$\bar{A}\bar{F} = \hat{E}^D \hat{A}\hat{E}^D \hat{E} = \hat{E}^D \hat{E}\hat{A}\hat{E}^D = \hat{E}^D \hat{E}\hat{E}^D \hat{A} = \bar{F}\bar{A} \tag{23}$$

and

$$\bar{F}x(t) = \hat{E}^D \hat{E} e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v = e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E}\hat{E}^D \hat{E} v = e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E} v = x(t) \tag{24}$$

since $\hat{E}^D \hat{E} e^{\hat{E}^D \hat{A} t} = e^{\hat{E}^D \hat{A} t} \hat{E}^D \hat{E}$ and $\hat{E}^D \hat{E}\hat{E}^D = \hat{E}^D$. □

**Definition 6** *The descriptor system* (9) *is called positive if* $x(t) \in \Re_+^n$, $t \geqslant 0$ *for all* $x(0) \in \Re_+^n$ *and* $x(0) \in \operatorname{im}(\bar{F})$.

**Theorem 7** *The descriptor continuous-time system* (9) *is positive if and only if*

$$\hat{E}^D \hat{A} \in M_n \text{ and } \operatorname{im}(\hat{E}^D \hat{E}) \in \Re_+^n. \tag{25}$$

**Proof** By Theorem 1 the standard continuous-time linear system (20a) is positive if and only if $\bar{A} \in M_n$. The descriptor system (11) and also (9) is positive if and only if the equivalent standard system (20a) is positive, i.e. (25) holds. □

**Example 1** *Consider the descriptor system* (9) *with the matrices*

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{26}$$

*The pencil of* (26) *is regular and for* $c = 1$ *we obtain*

$$[Ec - A]^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{27}$$

*and*

$$\begin{gathered} \hat{E} = [Ec - A]^{-1} E = \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \hat{A} = [Ec - A]^{-1} A = \begin{bmatrix} 0.5 & 0.25 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \end{gathered} \tag{28}$$

*Using one of the methods presented in* [13] *we compute*

$$\hat{E}^D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29}$$

*and*

$$\bar{A} = \hat{E}^D \hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{F} = \hat{E}^D \hat{E} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{30}$$

*By Theorem 7 the descriptor system* (11) *with* (28) *and also the system* (9) *with* (26) *is positive.*

### 3.2. Discrete-time system

Consider the descriptor autonomous discrete-time linear system

$$E'x_{i+1} = A'x_i, \quad i \in Z_+ = \{0, 1, ...\} \tag{31}$$

where $x_i \in \Re^n$ is the state vectors and $E', A' \in \Re^{n \times n}$.

It is assumed that rank $E' < n$ and the pencil $E'z - A'$ is regular, i.e.

$$\det[E'z - A'] \neq 0 \text{ for some } z \in C \text{ (the field of complex number).} \tag{32}$$

Choosing $z = c$ such that $\det[E'c - A'] \neq 0$ and pre-multiplying (31) by the matrix $[E'c - A']^{-1}$ we obtain

$$\hat{E}'x_{i+1} = \hat{A}'x_i \tag{33}$$

where

$$\hat{E}' = [E'c - A']^{-1}E'. \tag{34}$$

In a similar way as for (11) it can be shown that

$$\hat{E}'\hat{A}' = \hat{A}'\hat{E}'. \tag{35}$$

Lemma 1 with some evident modifications is also valid for the discrete-time systems.

**Theorem 8** [4,13] *The solution to the equation* (33) *has the form*

$$x_i = (\hat{E}'^D\hat{A}')^i\hat{E}'^D\hat{E}'v \tag{36}$$

*where $v \in \Re^n$ is arbitrary.*

Note that (36) is the solution of the difference equation

$$x_{i+1}\bar{A}'x_i \tag{37a}$$

where

$$\bar{A}' = \hat{E}'^D\hat{A}', \quad \bar{F}' = \hat{E}'^D\hat{E}', \quad x_i = (\bar{A}')^i\bar{F}'v. \tag{37b}$$

Therefore, the descriptor system described by (33) is equivalent to the standard system described by (37).

**Theorem 9** *For the standard system* (37) *the following holds*

$$\bar{A}'\bar{F}' = \bar{F}'\bar{A}' \tag{38}$$

*and*

$$x_i = \bar{F}'x_i, \quad i \in Z_+. \tag{39}$$

The proof is similar to the proof of Theorem 6.

**Definition 7** *The descriptor system* (31) *is called positive if* $x_i \in \Re^n_+$, $i \in Z_+$ *for all* $x_0 \in \Re^n_+$ *and* $x_0 \in \operatorname{im}(\bar{F}')$.

**Theorem 10** *The descriptor discrete-time system* (31) *is positive if and only if*

$$\hat{E}'^D \hat{A}' \in \Re^{n\times n}_+ \text{ and } \operatorname{im}(\hat{E}'^D \hat{E}') \in \Re^n_+. \tag{40}$$

The proof is similar to the proof of Theorem 7.

**Example 2** *Consider the descriptor system* (31) *with the matrices*

$$E' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 0 \end{bmatrix}, \quad A' = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix}. \tag{41}$$

*The pencil of* (41) *is regular and for* $c = 1$ *we obtain*

$$[E'c - A']^{-1} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 1 & 0.5 \\ 1 & 1 & 0.5 \\ 2 & 1 & 1 \end{bmatrix} \tag{42}$$

$$\begin{aligned} \hat{E}' &= [E'c - A']^{-1}E' = \begin{bmatrix} 3 & 1 & 0.5 \\ 1 & 1 & 0.5 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \hat{A}' &= [E'c - A']^{-1}A' = \begin{bmatrix} 3 & 1 & 0.5 \\ 1 & 1 & 0.5 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \end{aligned} \tag{43}$$

*Using one of the methods presented in* [13] *we compute*

$$\hat{E}'^D = \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{44}$$

*and*

$$\begin{aligned} \bar{A}' &= \hat{E}'^D \hat{A}' = \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ \bar{F}' &= \hat{E}'^D \hat{E}' = \begin{bmatrix} 0.5 & -0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{45}$$

By Theorem 10 the descriptor system (33) with (43) and also the system (31) with (41) is positive.

## 4. Asymptotic stability of positive descriptor linear systems

### 4.1. Continuous-time systems

**Definition 8** *The positive descriptor system* (11) *(and also* (9)*) is called asymptotically stable if*

$$\lim_{t\to\infty} x(t) = 0 \ \textit{for all} \ x(0) \in \mathrm{im}(\hat{E}^D\hat{E}) \ \textit{and} \ x(0) \in \Re^n_+. \tag{46}$$

**Theorem 11** *The positive descriptor system* (9) *is asymptotically stable if and only if all coefficients of the polynomial*

$$\det[I_n s - \hat{E}^D\hat{A}] = s^p(s^{n-p} + a_{n-p-1}s^{n-p-1} + \ldots + a_1 s + a_0) \tag{47}$$

*are positive, i.e.* $a_k > 0$ *for* $k = 0, 1, \ldots, n-p-1$ *where*

$$p = n - \mathrm{rank}\hat{E}^D\hat{A} \tag{48}$$

**Proof** By Theorem 2 the standard positive continuous-time system (1) is asymptotically stable if and only if all coefficients of the polynomial (4) are positive. From this fact and Definition 8 it follows that the positive descriptor system (9) is asymptotically stable if and only if all coefficients of the polynomial (47) are positive. □

**Example 3** (*Continuation of Example* 1) *In this case the matrix* $\hat{E}^D\hat{A}$ *has the form*

$$\hat{E}^D\hat{A} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{49}$$

*and its characteristic polynomial* (47) *is*

$$\det[I_3 s - \hat{E}^D\hat{A}] = \begin{vmatrix} s+1 & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{vmatrix} = s^2(s+1). \tag{50}$$

*The conditions of Theorem* 11 *are satisfied and the positive descriptor system with the matrices* (26) *is asymptotically stable.*

### 4.2. Discrete-time systems

**Definition 9** *The positive descriptor system* (31) *is called asymptotically stable if*

$$\lim_{i\to\infty} x_i = 0 \ \textit{for all} \ x_0 \in \mathrm{im}(\hat{E}'^D\hat{E}') \ \textit{and} \ x_0 \in \Re^n_+. \tag{51}$$

**Theorem 12** *The positive descriptor system* (31) *is asymptotically stable if and only if all coefficients of the polynomial*

$$\det[I_n(z+1)-\hat{E}'^D\hat{A}'] = (z+1)^{\bar{p}}(z^{n-\bar{p}}+\bar{a}_{n-p-1}z^{n-\bar{p}-1}+\ldots+\bar{a}_1 z+\bar{a}_0) \quad (52)$$

*are positive, i.e.* $\bar{a}_k > 0$ *for* $k = 0,1,\ldots,n-\bar{p}-1$*; where*

$$\bar{p} = n - \mathrm{rank}\hat{E}'^D\hat{A}' \quad (53)$$

**Proof** By Theorem 4 the standard positive discrete-time system (5) is asymptotically stable if and only if all coefficients of the polynomial (8) are positive. From this fact and Definition 9 it follows that the positive descriptor system (31) is asymptotically stable if and only if all coefficients of the polynomial (52) are positive. □

**Example 4** (*Continuation of Example* 2) *In this case the matrix* $\hat{E}'^D\hat{A}'$ *has the form*

$$\hat{E}'^D\hat{A}' = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad (54)$$

*and its characteristic polynomial* (52) *is*

$$\det[I_3(z+1)-\hat{E}'^D\hat{A}'] = \begin{vmatrix} z+0.5 & -0.5 & 0 \\ 0 & z+1 & 0 \\ 0 & 0 & z+1 \end{vmatrix} = (z+1)^2(z+0.5). \quad (55)$$

*The conditions of Theorem* 12*) are satisfied and the positive descriptor system with the matrices* (41) *is asymptotically stable.*

## 5. Extension to nonautonomous linear systems

### 5.1. Continuous-time system

Consider the descriptor continuous-time linear system

$$E\dot{x}(t) = Ax(t)+Bu(t), \quad x(0) = x_0 \quad (56)$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ are the state and input vectors and $E,A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$. It is assumed that and the condition (10) is satisfied.

**Theorem 13** [4,13] *The solution to the equation* (56) *has the form*

$$x(t) = e^{\hat{E}^D\hat{A}t}\hat{E}^D\hat{E}v + \int_0^t e^{\hat{E}^D\hat{A}(t-\tau)}\hat{E}^D\hat{B}u(\tau)d\tau + \sum_{k=0}^{q-1}(\hat{E}^D\hat{E}-I_n)(\hat{E}^D\hat{A}^D)^k\hat{A}^D\hat{B}u^{(k)}(t) \quad (57)$$

*where* $\hat{E},\hat{A}$ *are defined by* (12), $u^{(k)}(t) = \frac{d^k u(t)}{dt^k}$,

$$\hat{B} = [Ec - A]^{-1}B \tag{58}$$

*and* $v \in \Re^n$ *is arbitrary.*

**Definition 10** *The descriptor system* (56) *is called positive if* $x(t) \in \Re_+^n$, $t \geqslant 0$ *for all* $x(0) \in \Re_+^n$, $x(0) \in \mathrm{im}(\bar{F})$ *and all inputs* $u(t) \in \Re_+^m$, $t \geqslant 0$, $k = 0,1,\dots,q-1$.

**Theorem 14** *The descriptor continuous-time system* (56) *is positive if and only if*

$$\begin{gathered}\hat{E}^D\hat{A} \in M_n, \text{ and } \hat{E}^D\hat{B} \in \Re_+^{n\times m}, \ (\hat{E}^D\hat{E} - I_n)(\hat{E}^D\hat{A}^D)^k\hat{A}^D\hat{B} \in \Re_+^{n\times m} \\ \text{for } k = 0,1,\dots,q-1\end{gathered} \tag{59}$$

*and*

$$\mathrm{im}[H_0, H_1, \dots, H_q] \in \Re_+^n \tag{60}$$

*where*

$$H_k = \begin{cases} (\hat{E}^D\hat{E} - I_n)(\hat{E}^D\hat{A}^D)^k\hat{A}^D\hat{B} & \text{for} \quad k = 0,1,\dots,q-1 \\ \hat{E}^D\hat{E} & \text{for} \quad k = q. \end{cases} \tag{61}$$

**Proof** It is well-known [15] that if $E = I_n$ the standard continuous-time linear system (56) is positive if and only if $A \in M_n$ and $B \in \Re_+^{n\times m}$. Note that

$$\int_0^t e^{\hat{E}^D\hat{A}(t-\tau)}\hat{E}^D\hat{B}u(\tau)d\tau \in \Re_+^n \text{ for any } u(t) \in \Re_+^m, \ t \geqslant 0 \tag{62}$$

if and only if $\hat{E}^D\hat{A} \in M_n$ and $\hat{E}^D\hat{B} \in \Re_+^{n\times m}$. Therefore, the descriptor system (56) is positive if and only if the conditions (59) and (60) are satisfied. □

Theorem 11 with some evident modifications is also valid for the positive descriptor systems (56).

### 5.2. Discrete-time system

Consider the descriptor discrete-time linear system

$$E'x_{i+1} = A'x_i + B'u_i, \quad i \in Z_+ = \{0,1,\dots\} \tag{63}$$

where $x_i \in \Re^n$, ${}_i \in \Re^m$ are the state and input vectors and $E', A' \in \Re^{n\times n}$, $B' \in \Re^{n\times m}$. It is assumed that rank $E < n$ and the condition (32) is satisfied.

**Theorem 15** [4,13] *The solution to the equation* (63) *has the form*

$$x_i = (\hat{E}'^D\hat{A}')^i\hat{E}'^D\hat{E}'v + \sum_{k=0}^{i-1}(\hat{E}'^D\hat{A}')^{i-k-1}\hat{E}'^D\hat{B}'u_k$$

$$+\sum_{k=0}^{q-1}(\hat{E}'^D\hat{E}'-I_n)(\hat{E}'\hat{A}'^D)^k\hat{A}'\hat{B}'u_{i+k} \tag{64}$$

*where* $\hat{E}'$,$\hat{A}'$ *are defined by* (34),

$$\hat{B}'=[E'c-A']^{-1}B' \tag{65}$$

*and* $v\in\Re^n_+$ *is arbitrary.*

**Definition 11** *The descriptor system* (63) *is called positive if* $x_i\in\Re^n_+$, $i\in Z_+$ *for all* $x_0\in\Re^n_+$, $x_0\in\operatorname{im}(\bar{F}')$ *and all inputs* $u_i\in\Re^n_+$, $i\in Z_+$.

**Theorem 16** *The descriptor discrete-time system* (63) *is positive if and only if*

$$\begin{gathered}\hat{E}'^D\hat{A}'\in\Re^{n\times n}_+ \text{ and } \hat{E}'^D\hat{B}'\in\Re^{n\times m}_+,\ (\hat{E}'^D\hat{E}'-I_n)(\hat{E}'\hat{A}'^D)^k\hat{A}'\hat{B}'\in\Re^{n\times m}_+\\ \text{for } k=0,1,...,q-1\end{gathered} \tag{66}$$

*and*

$$\operatorname{im}[H'_0,H'_1,...,H'_q]\in\Re^n_+ \tag{67}$$

*where*

$$H'_k=\begin{cases}(\hat{E}'^D\hat{E}'-I_n)(\hat{E}'\hat{A}'^D)^k\hat{A}'\hat{B}' & \text{for} \quad k=0,1,...,q-1\\ \hat{E}'^D\hat{E}' & \text{for} \quad k=q.\end{cases} \tag{68}$$

**Proof** It is well-known [15] that if $E'=I_n$ the standard discrete-time linear system (63) is positive if and only if the conditions $\hat{A}'\in\Re^{n\times n}_+$ and $\hat{B}'\in\Re^{n\times m}_+$. Note that

$$\sum_{k=0}^{i-1}(\hat{E}'^D\hat{A}')^{i-k-1}\hat{E}'\hat{B}'u_k\in\Re^n_+ \text{ for all } u_i\in\Re^m_+,\ i\in Z_+ \tag{69}$$

if and only if $\hat{E}'^D\hat{A}'\in\Re^{n\times n}_+$ and $\hat{E}'^D\hat{B}'\in\Re^{n\times m}_+$. Therefore, the descriptor system (63) is positive if and only if the conditions (66) and (67) are satisfied. □

Theorem 12 with some evident modifications is also valid for the positive descriptor systems (63).

## 6. Concluding remarks

The positivity and asymptotic stability of the descriptor linear continuous-time and discrete-time systems with regular pencils have been addressed. Necessary and sufficient conditions for the positivity and asymptotic stability of the descriptor linear systems have been established. The applied method is based on the Drazin inverse matrix approach. The effectiveness of the proposed conditions has been demonstrated on numerical examples of the continuous-time and discrete-time systems. The considerations can be extended to the fractional descriptor linear systems and to 2D descriptor linear systems.

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