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A DISCONTINUOUS GALERKIN FINITE ELEMENT METHOD FOR DYNAMIC OF FULLY SATURATED SOIL

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The fully coupled, porous solid-fluid dynamic field equations with $u-p$ formulation are used in this paper to simulate pore fluid and solid skeleton responses. The present formulation uses physical damping, which dissipates energy by velocity proportional damping. The proposed damping model takes into account the interaction of pore fluid and solid skeleton.
The paper focuses on formulation and implementation of Time Discontinuous Galerkin (TDG) methods for soil dynamics in the case of fully saturated soil. This method uses both the displacements and velocities as basic unknowns and approximates them through piecewise linear functions which are continuous in space and discontinuous in time. This leads to stable and third-order accurate solution algorithms for ordinary differential equations. Numerical results using the time-discontinuous Galerkin FEM are compared with results using a conventional central difference, Houbolt, Wilson θ, HHT-α, and Newmark methods. This comparison reveals that the time-discontinuous Galerkin FEM is more stable and more accurate than these traditional methods.

Key words: Soil dynamics; discontinuous Galerkin finite element method; wave propagation; saturated porous soil.

1. Introduction

The equations governing the response of saturated porous media, incorporating the fluid-solid skeleton interaction, was first established for the quasi-static QS case in 1941 by Biot [1] who then extended them to include dynamics [2, 3]. Later, Truesdell [4, 5] introduced 'mixture theory' to formulate this problem, which provided a new basis for such coupled equations. Such formulations have been subsequently extended to consider the nonlinearity of deformation [6-8].

Multiphase ground deformations are caused by static loads resulting from dead load and temporary long-time loads. In addition, soil may transfer the short-time loads called dynamic loads, which depends on the dynamics characteristics: frequency and amplitudes. Types of loads acting on the soil are shown in Figure 1.

In QS analysis usual assumptions of drained or undrained behaviour are made depending on the rate of loading vis-à-vis the rate of drainage. In dynamic analysis,

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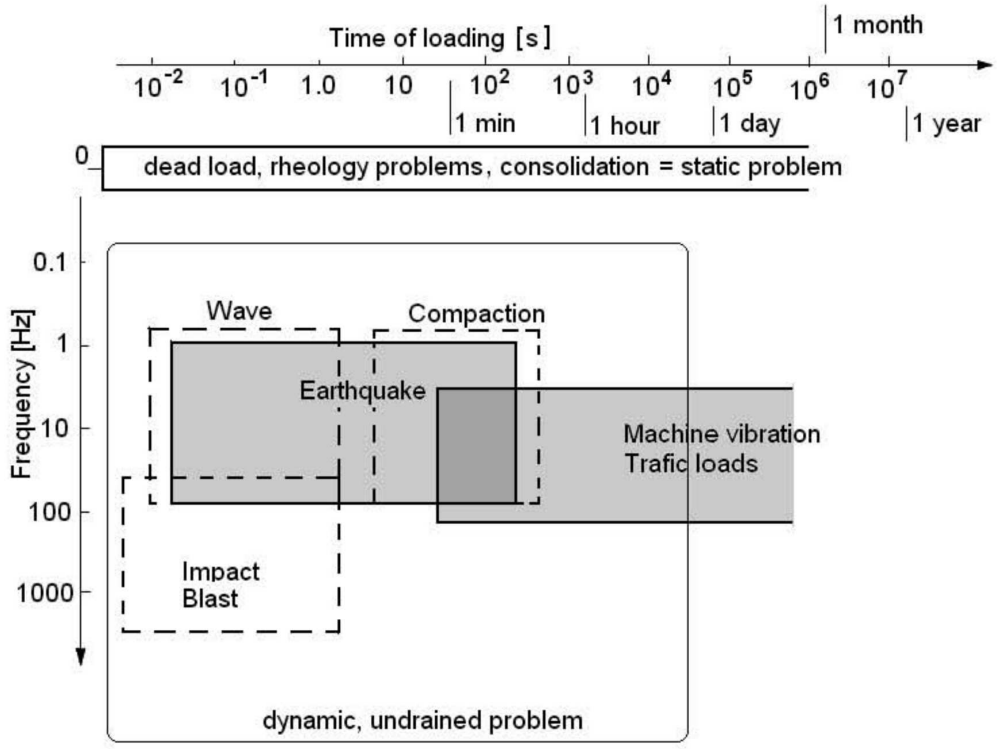

Fig. 1. Types of loads due to changes in time.
Rys. 1. Przypadki obciążenia zmiennego w czasie

the formulation is further complicated by the presence of inertial terms associated with both the motion of solid skeleton and that of the pore fluid. Depending on the rate of loading and the characteristics of flow and deformation, for the response of saturated porous media, the following three idealizations are possible:

- ***Fully dynamic* (*FD*):** In this case, the coupled equations of flow and deformation are formulated including the acceleration of both solid skeleton and fluid.
- ***Partly dynamic* (*PD*):** In this case, the coupled equations of flow and deformation consider only the acceleration of solid skeleton and not that of pore fluid. This is also called the $\mathbf{u}-p$ formulation as in this case the governing equations can be represented only in terms of solid displacement, $\mathbf{u}$ and pore fluid pressure, p.
- ***Quasi-static* (*QS*):** Here, all inertial terms are ignored resulting in QS coupled flow and deformation formulation.

The analytical solutions for various conditions and formulation are developed in [9] under plane strain condition with linear dynamic problem. Non-linear dynamic

problem with implicit Newmark's integration scheme in time domain can be found in [10].

2. Dynamic idealization

Depending on the motion of the pore fluid and the solid skeleton, as well as the permeability of the porous medium, it is possible to have different idealizations in the coupled flow and deformation problem. These idealizations were first described by Zienkiewicz *et al*. [11]. Considering the inertial forces associated with both solid skeleton and pore fluid, equations for ***FD*** are written as:

$$\sigma_{ij,j} + \rho\ddot{u}_i + \rho_{\mathrm{f}}\ddot{w}_i - \rho b_i = 0, \tag{2.1a}$$

$$-p_{,i} + \rho_{\mathrm{f}}\ddot{u}_i + \frac{\rho_{\mathrm{f}}}{n}\ddot{w}_i - \rho_{\mathrm{f}} b_i = \frac{\rho_{\mathrm{f}} g_i}{k_i}\dot{w}_i, \tag{2.1b}$$

$$\dot{\varepsilon}_{ii} - \dot{w}_{i,i} + \frac{n}{K_{\mathrm{f}}}\dot{p} = 0, \tag{2.1c}$$

where $\ddot{w}_i$ is the average fluid acceleration relative to solid skeleton, $\ddot{u}_i$ is the acceleration of solid skeleton, g_i is the appropriate component of the gravitational acceleration, ρ is the total density of the porous medium defined by $\rho = n\,\rho_f + (1-n)\,\rho_s$, with n denoting the porosity and ρ_f – the density of the fluid, and p – excess pore water pressure.

Ignoring the inertial forces associated with the relative pore fluid displacement ($\ddot{w}_i = 0$), the governing equations for ***PD*** become:

$$\sigma_{ij,j} + \rho\ddot{u}_i - \rho b_i = 0, \tag{2.2a}$$

$$-p_{,i} + \rho_{\mathrm{f}}\ddot{u}_i - \rho_{\mathrm{f}} b_i = \frac{\rho_{\mathrm{f}} g_i}{k_i}\dot{w}_i, \tag{2.2b}$$

$$\dot{\varepsilon}_{ii} - \dot{w}_{i,i} + \frac{n}{K_{\mathrm{f}}}\dot{p} = 0. \tag{2.2c}$$

The definition of the combined compressibility of the fluid and solid phases can be presented by

$$C = nC_{\mathrm{f}} + (\alpha - n)C_s \cong nC_{\mathrm{f}} + (1 - n)C_s, \tag{2.3}$$

where C_f is the compressibility of fluid defined by $C_f = 1/K_f$, and C_s is the compressibility of solid particles defined as $C_s = 1/K_s$. Substituting α and C into Eqs. (2.2c), leads to

$$\alpha\dot{\varepsilon}_{ii} - \dot{w}_{i,i} + \frac{\dot{p}}{C} = 0. \tag{2.4}$$

The pore fluid pressure p, the relative average velocity of fluid flow to solid phase $\dot{w}_i$, and the displacement of solid skeleton u_i are the unknown variables in these equations system. The variable $\dot{w}_i$ can be eliminated from the equations, so the simplified governing equation, which contain two independent variables u_i and p, can be achieved as follows:

$$\sigma_{ij,j} + \rho\ddot{u}_i - \rho b_i = 0. \tag{2.5}$$

Combining Eqs. (2.2b) and Eqs. (2.4) and substituting them into (2.4) we obtain the second governing equation

$$\alpha\dot{\varepsilon}_{ii} + \left[\frac{k_i}{\rho_{\mathrm{f}} g_i}\left(p_{,i} + \rho_{\mathrm{f}}\ddot{u}_i - \rho_{\mathrm{f}} b_i\right)\right]_{,i} + \frac{\dot{p}}{C} = 0. \tag{2.6}$$

Eqs. (2.5) and Eqs. (2.6) together form the $\mathbf{u} - p$ formulation, which must be solved in a coupled manner. In order to solve these equations, the initial and boundary conditions are necessary. For the total momentum balance on the part of boundary Γ_t, the total traction $\mathbf{t}$ is specified, while for Γ_u, the displacement $\mathbf{u}$ is given. For the fluid phase, the value of p is specified on Γ_p.

3. FE formulation of governing equations

The spatial discretization can be achieved by suitable shape functions for two variables u_i and p, defined as $\mathbf{u} = \mathbf{N}^u\bar{\mathbf{u}}$ and $p = \mathbf{N}^p\bar{\mathbf{p}}$, where $\mathbf{N}^u$ and $\mathbf{N}^p$ are the shape functions. The governing equations can now be transformed into a set of algebraic equations in space by the use of an appropriate Galerkin method. The discretization of first equation in space can be achieved by pre-multiplying Eqs. (2.5) by $(\mathbf{N}^u)^{\mathrm{T}}$ and by integrating it over the spatial domain, as

$$\mathbf{M}^u\ddot{\bar{\mathbf{u}}} + \int_\Omega \mathbf{B}^{\mathrm{T}}\sigma'\mathrm{d}\Omega - \alpha\mathbf{Q}\bar{\mathbf{p}} - \mathbf{f}^{(1)} = 0, \tag{3.1}$$

where the constitutive relation is taken into account and defined as $\delta\sigma' = \mathbf{D}\delta\varepsilon = \mathbf{DB}\delta\bar{\mathbf{u}}$, where $\mathbf{B}$ is the matrix relating to the increments of strain and displacements. In Eq. (3.1), the mass matrix $\mathbf{M}^u$, the coupling matrix $\mathbf{Q}$ and the load vector $\mathbf{f}^{(1)}$ are defined as

$$\mathbf{M}^u = \int_\Omega (\mathbf{N}^u)^{\mathrm{T}}\rho\mathbf{N}^u\mathrm{d}\Omega, \tag{3.2a}$$

$$\mathbf{Q} = \int_\Omega (\nabla\mathbf{N}^u)^{\mathrm{T}}\,\alpha\mathbf{m}\mathbf{N}^p\mathrm{d}\Omega, \tag{3.2b}$$

$$\mathbf{f}^{(1)} = \int_{\Omega} (\mathbf{N}^u)^{\mathrm{T}} \rho \mathbf{b} \mathrm{d}\Omega + \int_{\Gamma_t} (\mathbf{N}^u)^{\mathrm{T}} \bar{\mathbf{t}} \mathrm{d}\Gamma, \tag{3.2c}$$

where $\mathbf{m}^{\mathrm{T}} = \{1, 1, 1, 0, 0, 0\}$.

In a similar manner the second discretized equation is obtained by pre-multiplying Eqs. (2.6) by $(\mathbf{N}^p)^{\mathrm{T}}$ and integrating it over the spatial domain as

$$\mathbf{Q}\dot{\mathbf{u}} + \mathbf{H}\bar{\mathbf{p}} + \mathbf{G}\dot{\bar{\mathbf{p}}} - \mathbf{f}^{(2)} = 0, \tag{3.3}$$

where

$$\mathbf{H} = \int_{\Omega} (\nabla \mathbf{N}^p)^{\mathrm{T}} \mathbf{k} (\nabla \mathbf{N}^p) \, \mathrm{d}\Omega, \tag{3.4a}$$

$$\mathbf{G} = \int_{\Omega} (\mathbf{N}^p)^{\mathrm{T}} \frac{1}{C} \mathbf{N}^p \mathrm{d}\Omega, \tag{3.4b}$$

$$\mathbf{f}^{(2)} = -\int_{\Omega} (\mathbf{N}^p)^{\mathrm{T}} \nabla^{\mathrm{T}} (\mathbf{k} \rho_{\mathrm{f}} \mathbf{b}) \mathrm{d}\Omega + \int_{\Gamma_p} (\mathbf{N}^p)^{\mathrm{T}} \bar{\mathbf{q}} \mathrm{d}\Gamma. \tag{3.4c}$$

The overall system can be defined in terms of the variable set $\{\bar{\mathbf{u}}, \bar{\mathbf{p}}\}^{\mathrm{T}}$ as

$$\begin{vmatrix} \mathbf{M}^u & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{vmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{p}} \end{Bmatrix} + \begin{vmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{Q}^{\mathrm{T}} & \mathbf{G} \end{vmatrix} \begin{Bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{p}} \end{Bmatrix} + \begin{vmatrix} \mathbf{K}^{ep} & -\mathbf{Q} \\ \mathbf{0} & \mathbf{H} \end{vmatrix} \begin{Bmatrix} \bar{\mathbf{u}} \\ \bar{\mathbf{p}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}^{(1)} \\ \mathbf{f}^{(2)} \end{Bmatrix}, \tag{3.5}$$

where $\mathbf{K}^{ep} = \int_{\Omega} \nabla \mathbf{N}^{\mathrm{T}} \mathbf{D}^{ep} \nabla \mathbf{N} \mathrm{d}\Omega$.

In order to complete the numerical solution, it is necessary to integrate the differential Eqs. (3.5) in time

$$\mathbf{M}\ddot{\mathbf{d}}(t) + \mathbf{C}\dot{\mathbf{d}}(t) + \mathbf{K}\mathbf{d}(t) = \mathbf{f}(t), \tag{3.6}$$

where $\mathbf{d} = \{\bar{\mathbf{u}}, \bar{\mathbf{p}}\}^{\mathrm{T}}$

4. Hamilton's Law of Varying Action

Direct solutions to initial value problems, first proposed by Hamilton (1835) [4.1] include all of the forces acting on or within the system, whether conservative or nonconservative

$$\delta \int_{t_0}^{t_k} (T + W) dt = 0. \tag{4.1}$$

Bailey (1975, 1977) [13, 14] extended the direct solutions to conservative, nonconservative, stationary, or nonstationary systems with prescribed initial conditions, prescribed boundary conditions, and prescribed loading

$$\delta \int_{t_0}^{t_k} (T + W)dt - \sum\nolimits_{r=1}^{N} \frac{\partial T}{\partial \dot{q}_r} \delta q_r \Big|_{t_0}^{t_k} = 0, \tag{4.2}$$

where: $q_r(t)$ is the dependent variable representing the displacement (which may or may not be a generalized coordinate) of the r-th degree-of-freedom, $r = 1, 2, \ldots, N$;

T is the kinetic energy of the system;

$\delta W = \sum_r Q_r \delta q_r$ is the (real) evolutional work expression, which includes the work of the conservative, non-conservative and damping forces, where Q_r are the associated (generalized) forces.

Eq. (4.2), called as Hamilton's Law of Varying Action has been applied to linear, nonlinear, time invariant, and time varying dynamic systems in order to solve the response problem directly, without the use of differential equations of motion by: Baruch (1982) [15], Hodges (1991) [16], Öz(1995, 2010) [17, 18], Peters (1988) [19].

5. Time integration

In order to solve the equation of motion, the time-discontinuous Galerkin with discretisation in space and time is proposed:

$$\mathbf{M}\ddot{\mathbf{d}}(t) + \mathbf{C}\dot{\mathbf{d}}(t) + \mathbf{K}\mathbf{d}(t) = \mathbf{f}(t). \tag{5.1}$$

Various direct integration methods or step-by-step time integration methods have been widely used to obtain numerical solutions for Eq. (5.1) of elastodynamic or structural dynamic problems. Among these, the second-order accurate methods, such as the Houbolt method [20], the Newmark method [21], the Wilson- θ method [22], the Park method [23], and the HHT-α method [24], have been most frequently used.

The time-discontinuous Galerkin (TDG) method to the area of structural dynamics was first applied by Hughes and Hulbert [25-27].

The finite element discretization in Eq. (5.1) defines a damping matrix which takes into account physics of velocity dependent interaction of pore water and solid skeleton.

The finite element discretizations are used in both space and time simultaneously. The assumed nodal primary unknown vector and its derivative, with respect to time for the semi-discrete field equation, are independently interpolated by piecewise polynomial functions in the time domain. DGFEM permits discontinuities of functions at discretized time levels.

Let $0 = t_1 < \ldots < t_n < t_{n+1} < \ldots < t_{N+1} = T$ be a sequence of discrete time levels t_n with corresponding time steps $\Delta t_n = t_{n+1} - t_n$ and let the trial solution space be:

$$\mathbf{V}_n^h \subset \left\{ \mathbf{u} \in \mathbf{H}^1(S_n), \mathbf{u} = \bar{\mathbf{u}} \text{on} \Gamma_1 \times I \right\}, \tag{5.2}$$

and the weighting function space be:

$$\mathbf{W}_n^h \subset \left\{ \mathbf{w} \in \mathbf{H}^1 (S_n), \mathbf{w} = \mathbf{0} \text{on} \Gamma_1 \times I \right\}, \tag{5.3}$$

where for each space-time domain $S_n = \Omega \times I_n$, where $I_n = (t_n, t_{n+1})$.

We assume that both the space $\mathbf{V}_n^h$ and $\mathbf{W}_n^h$ consist of tensor products of bilinear functions of $\mathbf{x}$ and t on space-time elements. Moreover, the variables (displacements and velocities) are discretized both in space and in time.

We use the following notations to denote discontinuous functions at time t_n:

$$\mathbf{w}_n^+ = \lim_{\varepsilon \to 0^+} \mathbf{w}(t_n + \varepsilon), \tag{5.4a}$$

$$\mathbf{w}_n^- = \lim_{\varepsilon \to 0^-} \mathbf{w}(t_n - \varepsilon), \tag{5.4b}$$

where the functions are continuous over each time interval but allow jumps at the discrete time level t_n as shown in Fig. 2.

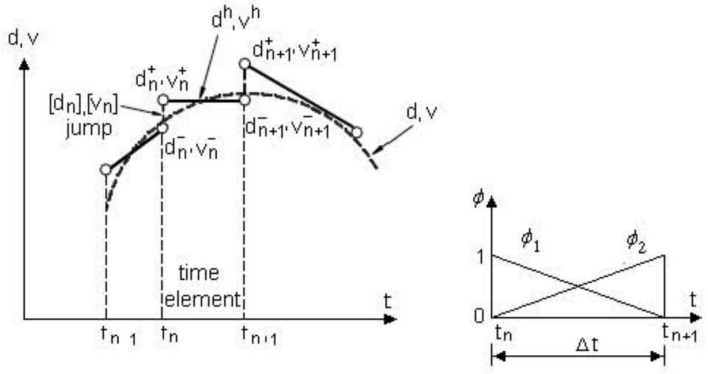

Fig. 2. Illustration of a) time-discontinuous approximation functions, b) linear shape function.
Rys. 2. Aproksymacja a) funkcją nieciągłą w czasie, b) liniowa funkcja kształtu

While considering a, let $\mathbf{d}_1$ and $\mathbf{v}_1$ denote the nodal displacements and velocities at t_n^+, respectively, and $\mathbf{u}_2$ and $\mathbf{v}_2$ the nodal displacements and velocities at t_{n+1}^-, respectively and time step $I_n = (t_n, t_{n+1})$. Also, let $\mathbf{d}_1^-$ and $\mathbf{v}_1^-$ represent the nodal displacements and velocities at t_n^-, respectively which are determined from either the previous steps calculations or, if n=1, as the initial data. Thus, the displacements and velocities at an arbitrary point $\mathbf{x}$ and time $t \in (t_n, t_{n+1})$ can be expressed as:

$$\mathbf{d}^h(\mathbf{x}, t) = \mathbf{N}(\mathbf{x})\phi_1(t)\mathbf{d}_1 + \mathbf{N}(\mathbf{x})\phi_2(t)\mathbf{d}_2, \tag{5.5}$$

where:

$$\phi_1(t) = \frac{t_{n+1} - t}{\Delta t}, \tag{5.5a}$$

(5.5b) $$\phi_2(t) = \frac{t - t_n}{\Delta t},$$

for P1-P1 as linear shape function.

The temporal derivative of the primary unknown vector, i.e. the velocity vector $\mathbf{v}(t)$, at arbitrary time $t \in [t_n, t_{n+1}]$ is interpolated as an independent variable by linear time shape functions as

(5.6) $$\mathbf{v}^h(\mathbf{x}, t) = \mathbf{N}(\mathbf{x})\phi_1(t)\mathbf{v}_1 + \mathbf{N}(\mathbf{x})\phi_2(t)\mathbf{v}_2.$$

As the vectors of the nodal displacements $\mathbf{d}$ and velocities $\mathbf{v}$ vary independently in the following variational equation in the time domain $I_n = \left(t_n^-, t_{n+1}^-\right)$, Equation (5.1) is re-expressed as

(5.7) $$\mathbf{M}\dot{\mathbf{v}} + \mathbf{C}\mathbf{v} + \mathbf{K}\mathbf{d} = \mathbf{f}^e,$$

with the constraint condition

(5.8) $$\dot{\mathbf{d}} - \mathbf{v} = \mathbf{0}.$$

The weak forms of the semi-discretized equation (5.7) and the constraint condition (5.8), together with the discontinuity conditions of $\mathbf{d}$ and $\mathbf{v}$ on a typical time sub-domain I_n can be expressed as

(5.9)
$$\int_{I_n} \delta\mathbf{v}^{\mathrm{T}} \left(\mathbf{M}\dot{\mathbf{v}} + \mathbf{C}\mathbf{v} + \mathbf{K}\mathbf{d} - \mathbf{f}^e\right) \mathrm{d}t + \int_{I_n} \delta\mathbf{d}^{\mathrm{T}}\mathbf{K}\left(\dot{\mathbf{d}} - \mathbf{v}\right) \mathrm{d}t + \delta\mathbf{d}_n^{\mathrm{T}}\mathbf{K}\left[[\mathbf{d}_n]\right] + \delta\mathbf{v}_n^{\mathrm{T}}\mathbf{M}\left[[\mathbf{v}_n]\right] = \mathbf{0}.$$

Substituting Eqs. (5.5) and (5.6) into Eq. (5.9), we obtain the following matrix equation from independent variations of $\mathbf{d}_n$, $\mathbf{d}_{n+1}$, $\mathbf{v}_n$, $\mathbf{v}_{n+1}$

(5.10)
$$\begin{vmatrix} \frac{1}{2}\mathbf{K} & \frac{1}{2}\mathbf{K} & -\frac{\Delta t}{3}\mathbf{K} & -\frac{\Delta t}{6}\mathbf{K} \\ -\frac{1}{2}\mathbf{K} & \frac{1}{2}\mathbf{K} & -\frac{\Delta t}{6}\mathbf{K} & -\frac{\Delta t}{3}\mathbf{K} \\ \frac{\Delta t}{3}\mathbf{K} & \frac{\Delta t}{6}\mathbf{K} & \frac{1}{2}\mathbf{M} + \frac{\Delta t}{3}\mathbf{C} & \frac{1}{2}\mathbf{M} + \frac{\Delta t}{6}\mathbf{C} - \frac{\Delta t^2}{12}\mathbf{K} \\ \frac{\Delta t}{6}\mathbf{K} & \frac{\Delta t}{3}\mathbf{K} & -\frac{1}{2}\mathbf{M} + \frac{\Delta t}{6}\mathbf{C} - \frac{\Delta t^2}{12}\mathbf{K} & \frac{1}{2}\mathbf{M} + \frac{\Delta t}{3}\mathbf{C} \end{vmatrix} \left\{ \begin{matrix} \mathbf{d}_n \\ \mathbf{d}_{n+1} \\ \mathbf{v}_n \\ \mathbf{v}_{n+1} \end{matrix} \right\} = \left\{ \begin{matrix} \mathbf{K}\mathbf{d}_n^- \\ \mathbf{0} \\ \mathbf{F}_1 + \mathbf{M}\mathbf{v}_n^- \\ \mathbf{F}_2 \end{matrix} \right\}$$

in which

(5.11a) $$\mathbf{F}_1 = \int_{I_n} \phi_1(t)\mathbf{F}\mathrm{d}t$$

$$\mathbf{F}_2 = \int_{I_n} \phi_2(t)\mathbf{F}\mathrm{d}t. \tag{5.11b}$$

This is the basic matrix equation of the time discontinuous Galerkin finite element method. The solutions for nodal displacement vectors $\mathbf{d}_n$, $\mathbf{d}_{n+1}$ are uncoupled from those for nodal velocity vectors $\mathbf{v}_n$, $\mathbf{v}_{n+1}$.

It is assumed that the nodal external force vector of the system varies within the incremental time step $I_n \in (t_n, t_{n+1})$ in the linear form, i.e.

$$\mathbf{F}(t) = \mathbf{f}(t_n)\phi_1 + \mathbf{f}(t_{n+1})\phi_2 = \mathbf{f}_n\phi_1 + \mathbf{f}_{n+1}\phi_2, \tag{5.12}$$

where $\mathbf{f}(t_n)$, $\mathbf{f}(t_{n+1})$ are the nodal external force vectors at times t_n, t_{n+1}. Substitution of expression (5.5) into expression (5.11) gives

$$\mathbf{F}_1 = \frac{\Delta t}{3}\mathbf{f}_n + \frac{\Delta t}{6}\mathbf{f}_{n+1}, \tag{5.13a}$$

$$\mathbf{F}_2 = \frac{\Delta t}{6}\mathbf{f}_n + \frac{\Delta t}{3}\mathbf{f}_{n+1}. \tag{5.13b}$$

Equation (5.10) can be recast as follows:

$$\begin{vmatrix} \frac{1}{2}\mathbf{K} & \frac{1}{2}\mathbf{K} & -\frac{\Delta t}{3}\mathbf{K} & -\frac{\Delta t}{6}\mathbf{K} \\ -\frac{1}{2}\mathbf{K} & \frac{1}{2}\mathbf{K} & -\frac{\Delta t}{6}\mathbf{K} & -\frac{\Delta t}{3}\mathbf{K} \\ \frac{\Delta t}{3}\mathbf{K} & \frac{\Delta t}{6}\mathbf{K} & \frac{1}{2}\mathbf{M} + \frac{\Delta t}{3}\mathbf{C} & \frac{1}{2}\mathbf{M} + \frac{\Delta t}{6}\mathbf{C} - \frac{\Delta t^2}{12}\mathbf{K} \\ \frac{\Delta t}{6}\mathbf{K} & \frac{\Delta t}{3}\mathbf{K} & -\frac{1}{2}\mathbf{M} + \frac{\Delta t}{6}\mathbf{C} - \frac{\Delta t^2}{12}\mathbf{K} & \frac{1}{2}\mathbf{M} + \frac{\Delta t}{3}\mathbf{C} \end{vmatrix} \begin{Bmatrix} \mathbf{d}_n \\ \mathbf{d}_{n+1} \\ \mathbf{v}_n \\ \mathbf{v}_{n+1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{K}\mathbf{d}_n^- \\ \mathbf{0} \\ \mathbf{F}_1 - \mathbf{F}_2 + \mathbf{M}\mathbf{v}_n^- \\ \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{M}\mathbf{v}_n^- - \Delta t\mathbf{K}\mathbf{d}_n^- \end{Bmatrix} \tag{5.14}$$

This is the basic matrix equation of the time discontinuous Galerkin finite element method. The solutions for nodal displacement vectors $\mathbf{d}_n$, $\mathbf{d}_{n+1}$ are uncoupled from those for nodal velocity vectors $\mathbf{v}_n$, $\mathbf{v}_{n+1}$. Equation (5.14) can be written as

$$\mathbf{d}_n = \mathbf{d}_n^-(\text{i.e.}\mathbf{d}_n^+ = \mathbf{d}_n^-), \tag{5.15}$$

(5.16)

$$\begin{vmatrix} \mathbf{M} + \frac{\Delta t}{6}\mathbf{C} - \frac{\Delta t^2}{12}\mathbf{K} & -\frac{\Delta t}{6}\mathbf{C} - \frac{\Delta t^2}{12}\mathbf{K} \\ \frac{\Delta t}{2}\mathbf{C} + \frac{\Delta t^2}{3}\mathbf{K} & \mathbf{M} + \frac{\Delta t}{2}\mathbf{C} + \frac{\Delta t^2}{6}\mathbf{K} \end{vmatrix} \begin{Bmatrix} \mathbf{v}_n \\ \mathbf{v}_{n+1} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_1 - \mathbf{F}_2 + \mathbf{M}\mathbf{v}_n^- \\ \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{M}\mathbf{v}_n^- - \Delta t\mathbf{K}\mathbf{d}_n^- \end{Bmatrix},$$

$$\mathbf{d}_{n+1} = \mathbf{d}_n^- + \frac{1}{2}\Delta t\,(\mathbf{v}_n + \mathbf{v}_{n+1})\,. \tag{5.17}$$

It should be noticed that continuity of the nodal displacement vector $\mathbf{d}_n$ at any time level t_n in the time domain $I = (0, T)$ is automatically ensured in the present DGFEM formulation. It is only the nodal velocity vectors at discretized time levels that remain discontinuous. Obviously, this is a significant advantage, particularly for materially non-linear problems, over the existing DGFEM formulations, in which both nodal displacements and velocities at both ends of a typical time step, i.e. at times t_n and t_{n+1}, are discontinuous.

6. Stability and accuracy

Stability and accuracy should be taken into account when considering the effectiveness of an iterative solution method in the time domain [28-31]

Stability analysis

In stability analysis, it is convenient to work with the undamped, free vibration single degree of freedom model problem

$$\ddot{d} + \omega^2 u = 0 \tag{6.1}$$

where ω denotes the undamped angular frequency.

Thus, Eq. (5.10) can be reduced to the simple form

$$\begin{vmatrix} \frac{1}{2}K & \frac{1}{2}K & -\frac{\Delta t}{3}K & -\frac{\Delta t}{6}K \\ -\frac{1}{2}K & \frac{1}{2}K & -\frac{\Delta t}{6}K & -\frac{\Delta t}{3}K \\ \frac{\Delta t}{3}K & \frac{\Delta t}{6}K & \frac{1}{2}M & \frac{1}{2}M \\ \frac{\Delta t}{6}K & \frac{\Delta t}{3}K & -\frac{1}{2}M & \frac{1}{2}M \end{vmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} Kd_1^- \\ 0 \\ Mv_1^- \\ 0 \end{Bmatrix}. \tag{6.2}$$

Herein, v_2 and d_2 can be determined in terms of d_1^- and v_1^- and then rearranged in a matrix form, as follows:

$$\begin{Bmatrix} v_2 \\ d_2 \end{Bmatrix} = \mathbf{A} \begin{Bmatrix} v_1^- \\ d_1^- \end{Bmatrix}, \tag{6.3}$$

where the matrix $\mathbf{A}$ denotes the amplification matrix. The explicit form is given as follows:

$$\mathbf{A} = \frac{1}{D} \begin{vmatrix} 36 - 14(\omega\Delta t)^2 & \left[-36(\omega\Delta t)^2 + 2(\omega\Delta t)^4\right]/\Delta t \\ \Delta t\left[36 - 2(\omega\Delta t)^2\right] & 36 - 14(\omega\Delta t)^2 \end{vmatrix}, \tag{6.4}$$

with $D = 36 + 4(\omega\Delta t)^2 + (\omega\Delta t)^4$.

The method is stable if each spectral radius satisfies the following condition, i.e.,

$$\rho(\mathbf{A}) \leq 1 \tag{6.5}$$

where the spectral radius ρ is defined as $\rho(\mathbf{A}) = \max(\lambda_1, \lambda_2)$.

The spectral radii of the amplification matrix are plotted in Fig. 3 for the P1-P1 TDG and commonly used direct integration methods. The TDG method succeeds in asymptotically annihilating spurious high-frequency behaviour without introducing excessive dissipation in the low-frequency regime.

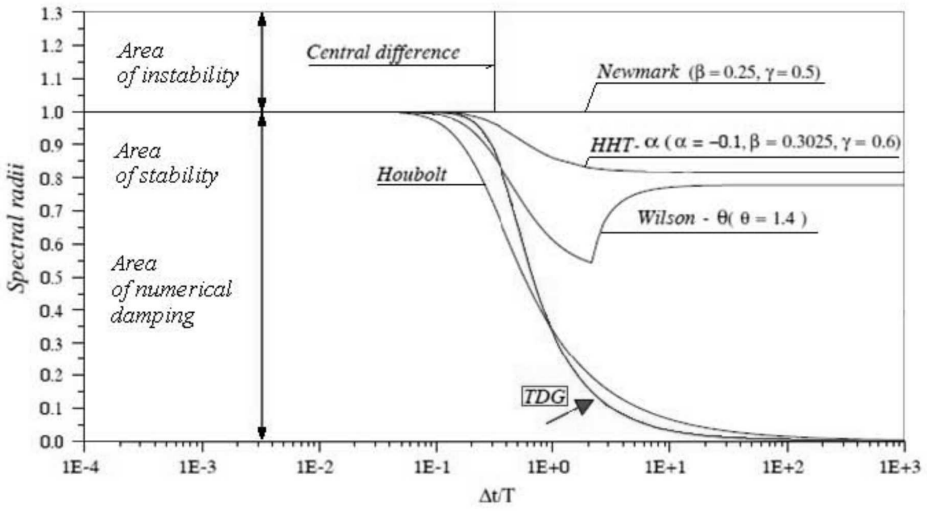

Fig. 3. Comparison of spectral radii for TDG, Central difference, Houbolt, Wilson-θ, HHT-α, and Newmark methods.

Rys. 3. Porównanie promienia spektralnego w algorytmach: TDG, różnicy centralnej, Houbolta, Wilsona-θ, HHT-α oraz metody Newmarka

Accuracy analysis

Accuracy refers to the difference between the numerical solution and the exact solution when the numerical solution process is stable. If the eigenvalues of **A** remain complex, i.e.,

$$\lambda_{1,2} = A \pm B\mathrm{i} = e^{(-\bar{\xi} \pm \mathrm{i})\bar{\Omega}}, \tag{6.6}$$

in which $B \neq 0$, and $\Omega = \omega\Delta t$ and $\bar{\Omega} = \bar{\omega}\Delta t$ ($\bar{\omega}$ is the approximate frequency evaluated from a numerical solution) the method considered here is stable.

From Eq. (6.6), we can obtain the following expression:

$$\bar{\Omega} = \tan^{-1}(B/A), \tag{6.7}$$

$$\bar{\xi} = \frac{-\ln(A^2 + B^2)}{2\bar{\Omega}}, \tag{6.8}$$

where $\bar{\xi}$, is the algorithmic damping ratio. The algorithmic damping ratio provides a measure of the numerical dissipation. The relative period error is taken as the measure of numerical dispersion and is calculated as

$$\frac{T - \bar{T}}{T} = \frac{\Omega}{\bar{\Omega}} - 1 = \frac{2\pi(\Delta t/T)}{\tan^{-1}(B/A)}, \tag{6.9}$$

where $T = 2\pi/\omega$ and $\bar{T} = 2\pi/\bar{\omega}$ are the exact natural period and the approximate natural period evaluated from a numerical solution, respectively.

The algorithmic damping ratios and the relative period errors are plotted and compared with other known methods in Figs. 4 and 5, respectively. Fig. 5 reveals very little numerical dissipation in the low-frequency regime. Meanwhile, Fig. 4 indicates that the period error of the TDG algorithm is virtually negligible in the low-frequency regime.

7. Conclusions and remarks

This work has presented a space-time finite element formulation that is implemented for the soil dynamics problems. The paper focuses on the formulation and implementation of Time Discontinuous Galerkin methods for soil dynamics. The algorithms derive directly from the implicit parent TDG method with piecewise linear functions in time (P1-P1), which approximate displacements and momenta.

Accuracy and stability analyses have been performed on undamped systems showing that the schemes are accurate and can be adopted with a user-defined dissipation, and are helpful in eliminating unresolved non-physical high-frequency modes in the time response. Moreover, the dissipative properties of the schemes are achieved without introducing any spurious root as only displacements and momenta are involved in the formulation.

Remarks:

1. Notes to the method of integration over time:

   a) The proposed method of integration over time provides the analysis of errors of approximation in time in terms of the Galerkin formulation of Hamilton principle

   b) For the load pulse, the efficiency of the discontinuous formulation is significantly higher compared to the classical methods

   c) The formulation described above includes the linear shape functions of P1. One can use higher degree polynomials, which increases the effectiveness of the method

2. TDG method permit assumed unknown vector and its derivative with respect to time to be discontinuous at the discrete time levels. It can effectively capture the

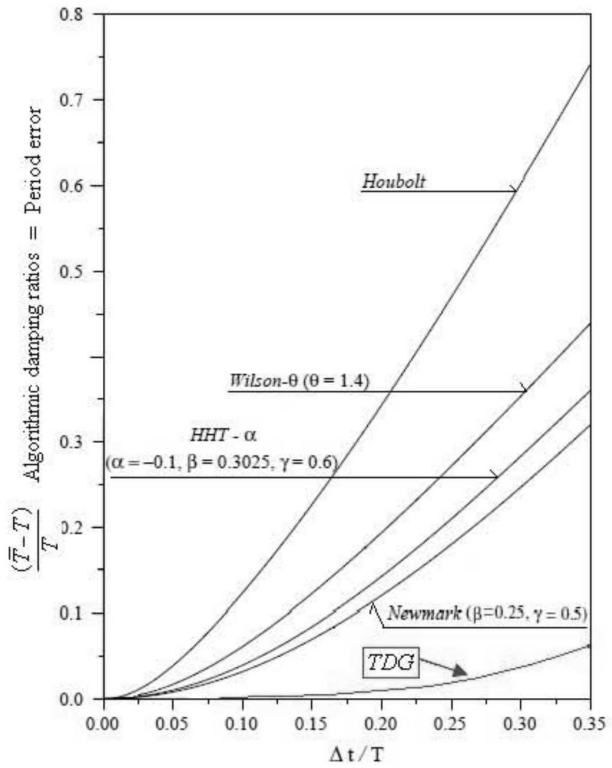

Fig. 4. Comparison of relative period errors for TDG, Houbolt, Wilson θ, HHT-α, and Newmark methods.

Rys. 4. Porównanie względnego błędu okresu w algorytmach: TDG, Houbolta, Wilsona-θ, HHT-α oraz metody Newmarka

discontinuities at the wave front and filter out the effects of spurious high modes and control spurious numerical oscillation.

3. The traditional Galerkin finite element method characterized by the semi-discrete procedure in spatial domain such as the Newmark method in time domain fails to capture discontinuities or sharp gradients of the solution for the dynamic problems subjected to impulse loads. In addition, it is also incapable of filtering out the effects of spurious high modes and controlling spurious numerical oscillation.
4. In TDG, continuity of the displacement vector at each discrete time instant is automatically ensured, whereas discontinuity of the velocity vector at the discrete time levels still remains. The computational cost is then obviously saved, particularly in the materially non-linear problems.

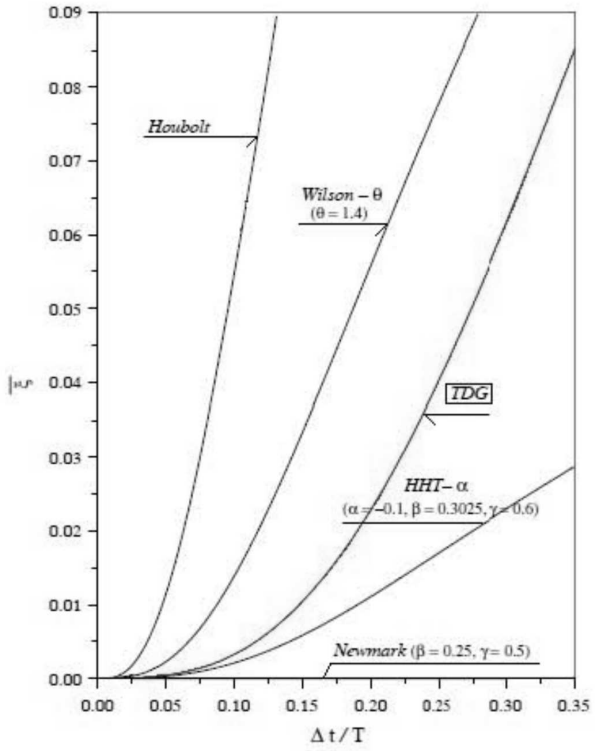

Fig. 5. Comparison of algorithmic damping ratio for TDG, Houbolt, Wilson θ, HHT-α, and Newmark methods.

Rys. 5. Porównanie procentu tłumienia krytycznego w algorytmach: TDG, Houbolta, Wilsona-θ, HHT-α oraz metody Newmarka

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ROZWIĄZANIE ZADANIA DYNAMIKI CAŁKOWICIE NAWODNIONEGO GRUNTU PRZY ZASTOSOWANIU MES Z NIECIĄGŁYM SFORMUŁOWANIEM GALERKINA W CZASIE

Streszczenie

Artykuł podejmuje zagadnienie analizy rozchodzenia się fal naprężeniowych w gruncie w ujęciu metody elementów skończonych bazując na sformułowaniu rozwiązania ciągłego w przestrzeni i nieciągłego w dziedzinie czasu Galerkina (*space and time-discontinous Galerkin* TDG *finite element method*). W tym sformułowaniu zarówno przemieszczenia jak i prędkości są wielkościami nieznanymi wzajemnie niezależnymi aproksymowanymi ciągłymi funkcjami kształtu w przestrzeni i nieciągłymi funkcjami kształtu w czasie.

Do opisu zachowania się gruntu w pełni nasyconego wodą zastosowano sformułowanie $u-p$ w ujęciu metody elementów skończonych. Grunt traktowany jest, jako ośrodek dwufazowy składający się ze szkieletu i wody w porach. Zastosowane sformułowanie uwzględnia tłumienie ośrodka przez uwzględnienie dyssypacji energii proporcjonalnej do prędkości wody względem szkieletu.

W artykule przedstawiono porównanie proponowanej metody rozwiązania numerycznego w dziedzinie czasu do metod obecnie stosowanych, takich jak: metoda różnicy centralnej, metoda Houbolta, Wilsona θ, HHT-α oraz najczęściej stosowanej metody Newmarka. Z porównania wynika, że proponowana metoda jest metodą stabilną o małym błędzie numerycznego rozwiązania.

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