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The digital function filters – algorithms and applications

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Abstract. The simple digital filters are not sufficient for digital modeling of systems with distributed parameters. It is necessary to apply more complex digital filters. In this work, a set of filters, called the digital function filters, is proposed. It consists of digital filters, which are obtained from causal and stable filters through some function transformation. In this paper, for several basic functions: exponential, logarithm, square root and the real power of input filter, the recursive algorithms of the digital function filters have been determined The digital function filters of exponential type can be obtained from direct recursive formulas. Whereas, the other function filters, such as the logarithm, the square root and the real power, require using the implicit recursive formulas. Some applications of the digital function filters for the analysis and synthesis of systems with lumped and distributed parameters (a long line, phase shifters, infinite ladder circuits) are given as well.

Key words: digital function filters, digital irrational filters, analysis and synthesis irrational filters, recursive algorithms of function filters.

1. Introduction

Currently, due to extensive use of DSP devices [1–3], the transition from the frequency domain to the time domain [4] is important, because circuit parameters can be determined based on the series of current and voltage samples by using digital filters without calculating the harmonics. However, in many situations - this concerns in particular the theory of systems with distributed parameters or the power theory, to model these cases digitally, simple digital rational filters are not sufficient. There is a need to use more complex digital filters. Such special filters, which include operators of simple digital filters as parameters, appear eg. as a result of solution of ordinary differential equations of long line [5, 6]. In the published literature there are studies on the logarithm filter (ie. cepstrum) [7], inverse filters are commonly found there in various solutions (these are simple recursions), and studies on the fractional order digital filters referring to differentiator and integrator filters [8–10]. In this paper, the set of filters, called the digital function filters, is proposed. This set consists of digital filters, obtained from linear, time-invariant, causal and stable filters by some function transformation.

Direct determination of the impulse response of the digital function filters by using the power series method is generally not possible because computational difficulties grow exponentially with an increasing sample number. Therefore, in this paper, for several basic functions, the recursive algorithms of the digital function filters have been formed. These functions are: exponential, logarithm, square root and the real power of input filter. Several applications of the digital function filters for the analysis and synthesis of systems with lumped and distributed parameters have been presented.

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2. The digital filters – recursive algorithms

Let $\boldsymbol{A}$ be a linear, time-invariant, causal and stable digital filter determined by the function:

$$A = A(z) = \sum_{n=0}^{\infty} A_n z^n \tag{1}$$

or equivalently by the series of weights (the impulse response):

$$A \leftrightarrow \{A_n\}, \quad n = 0, 1, 2, ... \tag{2}$$

The conditions are fulfilled:

$$A_n = 0 \quad \text{for} \quad n < 0, \tag{3}$$

$$\sum_{n=0}^{\infty} |A_n| < \infty \tag{4}$$

or equivalently

$$|A(z)| < \infty \quad \text{for} \quad z : |z| \leq 1. \tag{5}$$

The filter $\boldsymbol{A}$ after transformation $A(z) \rightarrow f(A(z))$ or in a brief notation

$$A \rightarrow f(A) \tag{6}$$

is called the function filter. It is also linear, time-invariant, causal and it is determined by the series of weights

$$\{(f(A))_n\}, n = 0, 1, 2, ... \tag{7}$$

or the function $(f(A))(z) = \sum_{n=0}^{\infty} (f(A))_n z^n$.

The purpose of this study is to formulate the recursive algorithms to obtain the series $\{(f(\boldsymbol{A}))_n\}$ from the series $\{\boldsymbol{A}_n\}$ for n= 0, 1, 2, . . .

Denoting by the symbol $(\cdot)' = d(\cdot)/dz$ the differentiation operation with respect to the variable "z", is obtained

$$z\,(f(A))' = zA'\frac{df}{dA}, \tag{8}$$

where df/dA is the digital filter determined by *ZET* function, or by the series of weights

$$\frac{df}{dA} = \frac{df}{dA}(z) \leftrightarrow \left\{\left(\frac{df}{dA}\right)_n\right\}; \quad n = 0, 1, 2, ... \tag{9}$$

The initial conditions are obtained from the formulas (6) and (8)

$$(f(A))_0 = f(A_0); \quad (f(A))_1 = A_1\frac{df}{dA}(0) \tag{10}$$

and formula (8) may become the recursive formula:

$$(f(A))_n = \sum_{m=1}^{n}\frac{m}{n}A_m\left(\frac{df}{dA}\right)_{n-m}. \tag{11}$$

Indeed, for the set of functions (set of the digital filters):

- the exponential type: e^{xA},
- the hyperbolical type:

$$\begin{aligned} chxA &= \frac{1}{2}\left(e^{xA} + e^{-xA}\right), \\ shxA &= \frac{1}{2}\left(e^{xA} - e^{-xA}\right), \end{aligned} \tag{12}$$

- and the elliptical type:

$$\begin{aligned} \cos xA &= \frac{1}{2}\left(e^{jxA} + e^{-jxA}\right), \\ \sin xA &= \frac{1}{2j}\left(e^{jxA} - e^{-jxA}\right), \end{aligned} \tag{13}$$

where x – a real parameter; $j = \sqrt{-1}$; A – given a linear, time-invariant, causal and stable digital filter of the series of weights $A \leftrightarrow \{A_n\}, n = 0, 1, 2, ...$; the derivative filters df/dA are the functions:

$$\frac{de^{xA}}{dA} = xe^{xA},$$

$$\frac{dchxA}{dA} = xshxA; \qquad \frac{dshxA}{dA} = xchxA,$$

$$\frac{d\cos xA}{dA} = -x\sin xA; \qquad \frac{d\sin xA}{dA} = x\cos xA.$$

Thus, formula (11) becomes the direct recursive formula for the functions:

$$e_n^{xA} = x\sum_{m=1}^{n}\frac{m}{n}A_m e_{n-m}^{xA}, \tag{14}$$

$$\begin{aligned} (chxA)_n &= x\sum_{m=1}^{n}\frac{m}{n}A_m(shxA)_{n-m}, \\ (shxA)_n &= x\sum_{m=1}^{n}\frac{m}{n}A_m(chxA)_{n-m} \end{aligned} \tag{15}$$

and

$$\begin{aligned} (\cos xA)_n &= -x\sum_{m=1}^{n}\frac{m}{n}A_m(\sin xA)_{n-m}, \\ (\sin xA)_n &= x\sum_{m=1}^{n}\frac{m}{n}A_m(\cos xA)_{n-m}. \end{aligned} \tag{16}$$

The initial conditions for the formula (14) and for the "cross" formulas (15) and (16) are obtained directly from (10):

$$e_0^{xA} = e^{xA_0}, \tag{17}$$

$$\begin{aligned} (chxA)_0 &= chxA_0, \\ (shxA)_0 &= shxA_0, \end{aligned} \tag{18}$$

$$\begin{aligned} (\cos xA)_0 &= \cos xA_0, \\ (\sin xA)_0 &= \sin xA_0. \end{aligned} \tag{19}$$

Further elements of the series of weights for the filters of exponential, hyperbolical and elliptical type, are determined from the formulas (14), (15) and (16) for $n = 1, 2, \ldots$

3. Implicit recursive algorithms

For the function $f(A)$ as *ln A* and $\sqrt{A}$, formula (8) does not lead directly to the recursive formulas. For *ln A* the derivative is:

$$\frac{d\ln A}{dA} = A^{-1}$$

and after applying it to the formula (8)

$$z(\ln A)'A = zA' \qquad \text{is obtained,}$$

hence

$$\sum_{m=1}^{n} m(\ln A)_m A_{n-m} = nA_n$$

or

$$(\ln A)_n = \frac{A_n}{A_0} - \frac{1}{A_0}\sum_{m=1}^{n-1}\frac{m}{n}(\ln A)_m A_{n-m}. \tag{20}$$

Expression (20) is already the recursive formula with initial conditions resulting from (10):

$$(\ln A)_0 = \ln A_0; \qquad (\ln A)_1 = \frac{A_1}{A_0}. \tag{21}$$

The algorithm of the "*logarithm*" can also be obtained otherwise, writing formula (8) as

$$z(\ln A)' = zA'A^{-1}$$

or for samples

$$(\ln A)_n = \sum_{m=1}^{n}\frac{m}{n}A_m A_{n-m}^{-1}. \tag{22}$$

Formula (22) can be used with an inversion of the filter A, ie:

$$AA^{-1} = 1$$

or

$$\sum_{m=0}^{n} A_m A_{n-m}^{-1} = \delta_n = \begin{cases} 0 n \neq 0 \\ 1 n = 0 \end{cases}$$

hence

$$A_0^{-1} = (A_0)^{-1},$$
$$A_n^{-1} = -\frac{1}{A_0}\sum_{m=1}^{n} A_m A_{n-m}^{-1}.$$

Using formula (8) to the filter $\sqrt{A}$ leads to the formula:

$$z(\sqrt{A})' = \frac{1}{2} zA'(\sqrt{A})^{-1}$$

or

$$z(\sqrt{A})'\sqrt{A} = \frac{1}{2} zA'$$

hence

$$\sum_{m=1}^{n} m(\sqrt{A})_m(\sqrt{A})_{n-m} = \frac{1}{2} nA_n.$$

After separation of this formula, the equation is obtained:

$$\sqrt{A_0} n(\sqrt{A})_n = \frac{1}{2} nA_n - \sum_{m=1}^{n-1} m(\sqrt{A})_m(\sqrt{A})_{n-m}$$

which takes the form of recursion:

$$(\sqrt{A})_n = \frac{1}{2}\frac{A_n}{\sqrt{A_0}} - \frac{1}{\sqrt{A_0}}\sum_{m=1}^{n-1}\frac{m}{n}(\sqrt{A})_m(\sqrt{A})_{n-m}. \tag{23}$$

Square root of the operator A can also be obtained by the direct method:

$$\sqrt{A}\sqrt{A} = A$$

or from the convolutional equation:

$$\sum_{m=0}^{n} (\sqrt{A})_m(\sqrt{A})_{n-m} = A_n$$

which, after separation of the components:

$$2(\sqrt{A})_n\sqrt{A_0} = A_n - \sum_{m=1}^{n-1} (\sqrt{A})_m(\sqrt{A})_{n-m}$$

takes the form of recursive equation:

$$(\sqrt{A})_n = \frac{A_n}{2\sqrt{A_0}} - \frac{1}{2\sqrt{A_0}}\sum_{m=1}^{n-1} (\sqrt{A})_m(\sqrt{A})_{n-m}. \tag{24}$$

The recursive formulas (23) and (24) differ by the component

$$\sum_{m=1}^{n-1}\left(\frac{m}{n} - \frac{1}{2}\right) x_m x_{n-m} = \sum_{m=1}^{n-1}\frac{m-(n-m)}{2n} x_m x_{n-m}$$

which, however, disappears because of antisymmetry when replacing $n-m \leftrightarrow m$.

4. The real power of the digital filter operator

The following function is considered

$$A \to A^p,$$

where p – real number.

This task can be solved by two methods:

- the direct method,
- the logarithm method (two steps).

The direct method comes from the formula (8):

$$z(A^p)' = pzA'A^{p-1} \tag{25}$$

or

$$z(A^p)'A = pzA'A^p.$$

Hence the convolutional equation is obtained:

$$\sum_{m=0}^{n} m(A^p)_m A_{n-m} = p\sum_{m=0}^{n} mA_m A_{n-m}^p$$

which after separation of the components takes the form:

$$nA_n^p A_0 = p\sum_{m=1}^{n} mA_m A_{n-m}^p - \sum_{m=1}^{n-1} mA_m^p A_{n-m}^p$$
$$= pnA_n A_0^p + \sum_{m=1}^{n-1} m(pA_m A_{n-m}^p - A_m^p A_{n-m})$$

and turns into the recursive equation:

$$A_n^p = p(A_0)^{p-1}A_n + \frac{1}{A_0}\sum_{m=1}^{n-1}\frac{m}{n}(pA_n A_{n-m}^p - A_m^p A_{n-m}). \tag{26}$$

Equation (25) can also be written as

$$z\frac{(A^p)'}{A^p}A = pzA'$$

or

$$z(\ln A^p)'A = pzA' \tag{27}$$

which leads to the "*logarithm method*". Equation (27) should be written in the convolutional equation form so that the series $\{(lnA^p)_n\}$ could be determined, ie:

$$\sum_{m=0}^{n} m(\ln A^p)_m A_{n-m} = pnA_n$$

or

$$n(\ln A^p)_n A_0 = pnA_n - \sum_{m=1}^{n-1} m(\ln A^p)_m A_{n-m}$$

hence

$$(\ln A^p)_n = p\frac{A_n}{A_0} - \frac{1}{A_0}\sum_{m=1}^{n-1}\frac{m}{n}(\ln A^p)_m A_{n-m}. \tag{28}$$

Applying the formula:

$$A^p = e^{\ln A^p}$$

and the recursion (14), is obtained

$$A_n^p = \sum_{m=1}^{n}\frac{m}{n}(\ln A^p)_m A_{n-m}^p. \tag{29}$$

The initial conditions for the series (26), (28) and (29) are obtained from the formulas

$$A_0^p = (A_0)^p.$$
$$(\ln A^p)_0 = \ln(A_0)^p = p\ln A_0.$$

5. Applications

The differential equation of a long line in partial derivatives has the form:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + 2\beta\frac{\partial u}{\partial t} + RGu = 0 \tag{30}$$

for line: R, L, G, $C > 0$, and

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + 2\beta\frac{\partial u}{\partial t} + RGu = 0 \tag{31}$$

for line: R, L, $-G$, $-C$.

Equation (30) is the wave equation, whereas (31) is the Laplace equation. Digital – operator models after the substitution:

$$\frac{\partial}{\partial t} \to s \to \frac{1}{\tau}(1-z). \tag{32}$$

Equations (30) and (31) turn into the ordinary derivative equations with operators:

$$\frac{d^2 u}{dx^2} - A^2 u = 0 \tag{33}$$

for the equation (30), and

$$\frac{d^2 u}{dx^2} + A^2 u = 0 \tag{34}$$

for the equation (31).

The following symbols were introduced:

$$t \to \frac{t}{\sqrt{LC}}; \qquad \beta = \frac{R\rho^{-1} + G\rho}{2}; \qquad \rho = \sqrt{\frac{L}{C}} \tag{35}$$

A is the square root operator:

$$A = (s^2 + 2\beta s + RG)^{\frac{1}{2}} = \left((s + R\rho^{-1})(s + G\rho)\right)^{\frac{1}{2}} = \frac{1}{\tau}\left((a-z)(b-z)\right)^{\frac{1}{2}}, \tag{36}$$

where $a = 1 + \tau R\rho^{-1}$; $b = 1 + \tau G\rho$.

For a simple operator, the expansion of real p-th power can be obtained directly from the formula:

$$(a-z)^p \leftrightarrow \left\{k_n a^p a^{-n}\right\}_{n=0,1,2,\ldots}, \tag{37}$$

where $\{kn\}$ – the universal series defined by the recursive formula:

$$k_n = \frac{n-1-p}{n}k_{n-1}; \qquad k_0 = 1. \tag{38}$$

For the complex operators, the recursive formulas given in Sec. 4 should be applied.

Whereas, the samples expansion of the operator A can be obtained by convolution:

$$A_n = \sqrt{ab}a^{-n}\sum_{m=0}^{n}\left(\frac{a}{b}\right)^m k_{n-m}k_m, \tag{39}$$

where

$$k_m = \frac{2m-3}{2m}k_{m-1}; \qquad k_0 = 1. \tag{40}$$

The general solution of the differential equation (33) has the form:

$$u^x = (chxA)p + (shxA)q, \tag{41}$$

where p, q – any time-variable signals with the samples expansion in the series form $\{pn\}$, $\{qn\}$, n = 0,1,2, ...

The particular solution satisfying the boundary conditions:

$$u^{x=0} = u^0; \qquad u^{x=l} = u^l \tag{42}$$

at the beginning ($x = 0$) and at the point $x = l$ is the:

$$u^x = \frac{sh(l-x)A}{shlA}u^0 + \frac{shxA}{shlA}u^l. \tag{43}$$

Equation (43) in samples notation is a convolution:

$$u_n^x = \sum_{m=0}^{n}\left(\frac{sh(l-x)A}{shlA}\right)_{n-m}u_m^0 + \sum_{m=0}^{n}\left(\frac{shxA}{shlA}\right)_{n-m}u_m^l, \tag{44}$$

where the partial convolutions take the form:

$$\left(\frac{shxA}{shlA}\right)_n = \sum_{m=0}^{n}(shxA)_{n-m}(shlA)_m^{-1} \tag{45}$$

and inversion of the *shlA* filter is determined from the recursive formulas:

$$\begin{aligned} (shlA)_0^{-1} &= (shlA_0)^{-1} \qquad \text{for} \quad n = 0, \\ (shlA)_n^{-1} &= -\frac{1}{shlA_0}\sum_{m=1}^{n}(shlA)_m(shlA)_{n-m}^{-1} \\ &\text{for} \quad n = 1,2,3,\ldots \end{aligned} \tag{46}$$

While, the general solution of the differential equation (34) is the operator:

$$u^x = (\cos xA)p + (\sin xA)q \tag{47}$$

acting on any signals p and q.

The particular solution satisfying boundary conditions:

$$u^{x=0} = u^0; \qquad u^{x=l} = u^l$$

takes the form:

$$u^x = \frac{\sin(l-x)A}{\sin lA}u^0 + \frac{\sin xA}{\sin lA}u^l. \tag{48}$$

As an example of a filter "*p-th power*" of the operator A, the phase shifter can be used adjusted in exponential way. The transmittance of such a system, in analog form, is:

$$\left(\frac{\alpha - s}{\alpha + s}\right)^p,$$

where $0 \le p \le 1$, α – positive real number.

The frequency response is an allpass type with the phase correction:

$$\left(\frac{\alpha - j\omega}{\alpha + j\omega}\right)^p = e^{-j2parctg\frac{\omega}{\alpha}}$$

and hence (see chart in Fig. 1)

$$\varphi(\omega) = -2parctg\frac{\omega}{\alpha}.$$

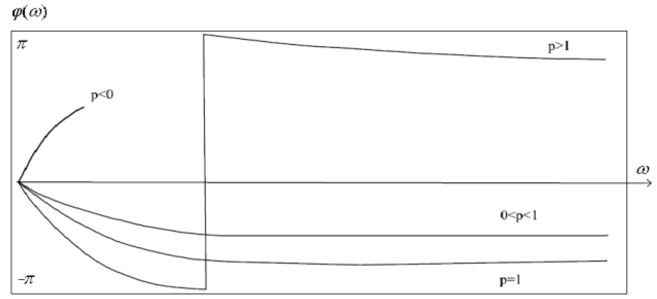

Fig. 1. Phase response of the filter

Digital model of phase shifter with bilinear transform has the form

$$\left(\frac{\alpha - s}{\alpha + s}\right)^p \rightarrow \frac{\alpha - \frac{2}{\tau}\dfrac{1-z}{1+z}}{\alpha + \dfrac{2}{\tau}\dfrac{1-z}{1+z}} = a\frac{a^{-1}+z}{a+z}$$

hence

$$\left(\frac{\alpha - s}{\alpha + s}\right)^p = a^p \frac{(a^{-1}+z)^p}{(a+z)^p},$$

where

$$a = \frac{\alpha\tau + 2}{\alpha\tau - 2}.$$

However, the resulting digital filter does not meet the root condition [5]:

$$A(z) > 0 \quad \text{for} \quad z \in \{z : |z| \leq 1\} \cap \mathbf{R},$$

R – set of real numbers.

However, the root condition is met by another filter with "analog" transmittance:

$$\left(\frac{a+s}{b+s}\right)^p,$$

where a, b – the positive real number.

The frequency response of this filter is:

$$\left(\frac{a+j\omega}{b+j\omega}\right)^p = \left(\frac{a^2+\omega^{\ 2}}{b^2+\omega^2}\right)^{\frac{p}{2}} e^{jp(arctg\frac{\omega}{a} - arctg\frac{a}{b})}.$$

Denoting by $K(\omega)$ the module response (the amplitude response) and by $\varphi(\omega)$ – phase response, is obtained:

$$K(\omega) = \left(\frac{a^2+\omega^{\ 2}}{b^2+\omega^2}\right)^{\frac{p}{2}},$$

$$\varphi(\omega)\varphi = p(arctg\frac{\omega}{a} - arctg\frac{a}{b}).$$

There is

$$K(0) = \left(\frac{a}{b}\right)^p; \qquad K(\infty) = 1$$

and also:

$$\frac{d\varphi}{d\omega} = p\left(\frac{\dfrac{1}{a}}{1+\dfrac{\omega^2}{a^2}} - \frac{\dfrac{1}{b}}{1+\dfrac{\omega^2}{b^2}}\right) = 0.$$

The condition

$$\frac{d\varphi}{d\omega} = 0$$

turns to the equation

$$\frac{1}{a}\left(1+\frac{\omega^2}{b^2}\right) - \frac{1}{b}\left(1+\frac{\omega^2}{a^2}\right) = 0$$

hence

$$\omega = \sqrt{ab}.$$

The maximum value of phase response is:

$$\varphi_{\max} = p\left(arctg\sqrt{\frac{b}{a}} - arctg\sqrt{\frac{a}{b}}\right).$$

In Fig. 2 the phase and amplitude responses were shown.

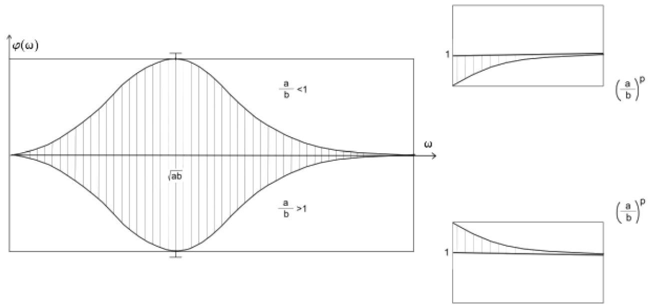

Fig. 2. Phase and amplitude responses of the phase shifter

The analysis of the graphs in Fig. 2 shows that, it is possible to continuously adjust the maximum point of phase response.

Some other properties have the allpass phase shifter with "analog" transmittance:

$$\left[\frac{(a-s)(a^*-s)}{(a+s)(a^*+s)}\right]^p = \left[\frac{aa^* - 2(Re\ a)s + s^2}{aa^* + 2(Re\ a)s + s^2}\right]^p$$

and the frequency response

$$\left[\frac{(aa^*-\omega^2) - j2(Re\ a)\omega}{(aa^*-\omega^2) + j2(Re\ a)\omega}\right]^p = e^{-j2parctg\frac{(2Re\ a)\omega}{aa^*-\omega^2}}.$$

In Fig. 3 phase response of filter was shown.

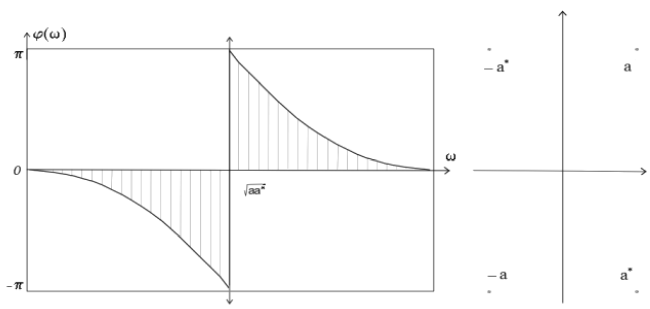

Fig. 3. Phase response of allpass filter and the distribution of its zeros and poles

Using the transformation $s \rightarrow \dfrac{1}{\tau}(1-z)$ to digital simulation of the filter is obtained

$$\frac{(a-s)(a^*-s)}{(a+s)(a^*+s)} = \frac{(1-a\tau-z)(1-a^*\tau-z)}{(1+a\tau-z)(1+a^*\tau-z)} = \frac{(\alpha-z)(\alpha^*-z)}{(\beta-z)(\beta^*-z)},$$

where

$$\alpha = 1 - a\tau; \qquad \beta = 1 + a\tau$$

and using the bilinear transform:

$$s \to \frac{2}{\tau}\frac{1-z}{1+z}$$

the function is obtained:

$$\sigma\sigma^* \frac{\left(\frac{1}{\sigma} - z\right)\left(\frac{1}{\sigma^*} - z\right)}{(\sigma - z)(\sigma^* - z)} = \frac{1 - (\sigma + \sigma^*)z + \sigma\sigma^* z^2}{\sigma\sigma^* - (\sigma + \sigma^*)z + z^2},$$

where

$$\sigma = \frac{2 + a\tau}{2 - a\tau}.$$

Another example of the application of the integral - derivative fractional order complex operators is an infinite ladder circuit. A general diagram of such homogeneous circuit in Fig. 4 is shown

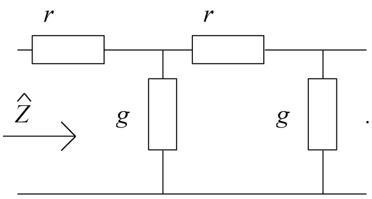

Fig. 4. Diagram of infinite homogeneous ladder circuits with operators: a horizontal r and a vertical g

This circuit consists of two operators: horizontal – impedance type (r) and vertical – admittance type (g).

The input impedance operator meets the following recursive formula:

$$\widehat{Z}_{n+1} = r + \cfrac{1}{g + \cfrac{1}{\widehat{Z}_n}}$$

which implies the boundary impedance equation of infinite ladder circuit

$$\widehat{Z} = r + \cfrac{1}{g + \cfrac{1}{\widehat{Z}}}$$

or

$$g\widehat{Z}^2 - rg\widehat{Z} - r = 0. \tag{49}$$

The solution of equation (49) is

$$\widehat{Z} = \frac{1}{2}r + \frac{1}{2}\sqrt{\frac{r}{g}}\sqrt{4 + rg}. \tag{50}$$

For electric ladder RL, GC, that simulates the classic long line (see Fig. 5) the operators r and g take the form of *PD*-type operators:

$$r \to R + sL; \qquad g \to G + sC$$

hence

$$\widehat{Z}(s) = \frac{1}{2}L\left[(a+s) + \frac{(a+s)^{\frac{1}{2}}}{(b+s)^{\frac{1}{2}}} (s^2 + (a+b)s + ab + 4\omega^2)^{\frac{1}{2}}\right] \tag{51}$$

where

$$a = \frac{R}{L}; \qquad b = \frac{G}{C}; \qquad \omega^2 = \frac{1}{LC}. \tag{52}$$

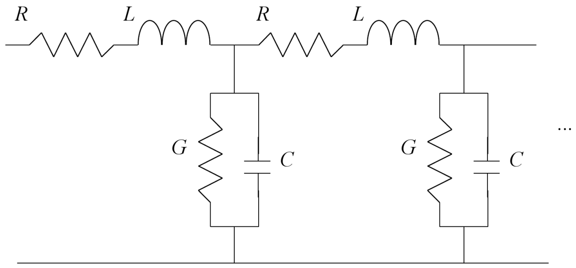

Fig. 5. The infinite homogeneous electric ladder RL, GC

The second degree polynomial appearing in the expression (51) has a distribution with respect to the pair of complex conjugate zeros:

$$s^2 + (a+b)s + ab + 4\omega^2 = (\sigma + s)(\sigma^* + s), \tag{53}$$

where

$$\sigma = \frac{a+b}{2} + j\sqrt{(2\omega)^2 - \left(\frac{a-b}{2}\right)^2}$$

and therefore the function (51) takes the form:

$$\widehat{Z}(s) = \frac{1}{2}L(a+s)^{\frac{1}{2}}(b+s)^{\frac{1}{2}} \frac{(\sigma+s)^{\frac{1}{2}}(\sigma^*+s)^{\frac{1}{2}} + (a+s)^{\frac{1}{2}}(b+s)^{\frac{1}{2}}}{b+s}. \tag{54}$$

Digital modeling of expression (54) gives:

$$(a+s)^{\frac{1}{2}} \to \left(a + \frac{1}{\tau}(1-z)\right)^{\frac{1}{2}} = \frac{1}{\sqrt{\tau}}(1 + a\tau - z)^{\frac{1}{2}}$$

hence

$$\widehat{Z} = \frac{1}{2}\frac{L}{a\tau}(a-z)^{\frac{1}{2}}(b-z)^{\frac{1}{2}} \frac{(\sigma-z)^{\frac{1}{2}}(\sigma^*-z)^{\frac{1}{2}} + (a-z)^{\frac{1}{2}}(b-z)^{\frac{1}{2}}}{b-z}, \tag{55}$$

where

$$\begin{gathered} a \to 1 + a\tau > 1, \\ b \to 1 + b\tau > 1, \\ \sigma \to 1 + \sigma\tau > 1, \qquad |\sigma| > 1. \end{gathered} \tag{56}$$

Equation (55) is significant in the occurrence of complex conjugate zeros. A variant of formulas (37), (38) for the complex conjugate zeros gives

$$\begin{aligned} &(\sigma - z)^p(\sigma^* - z)^p \leftrightarrow \\ &\sum_{m=0}^{n} k_{n-m}k_m\sigma^p\sigma^{-(n-m)}(\sigma^*)^p(\sigma^*)^{-m} \\ &= \sum_{m=0}^{n} |\sigma|^{2p} k_{n-m}k_m Re\left[\sigma^{-n}\left(\frac{\sigma}{\sigma^*}\right)^m\right] \\ &= |\sigma|^{2p-n}\sum_{m=0}^{n} k_{n-m}k_m \cos\left(|(n-m) - m| < \sigma\right). \end{aligned} \tag{57}$$

6. Summary

In this paper the methods to determine recursive algorithms of the digital function filters (the irrational filters) have been presented. The digital function filters of exponential, hyperbolical and elliptical type can be obtained by direct recursive formulas. For the other function filters, such as the logarithm and the real power, implicit recursive formulas are required. The function filters of exponential, hyperbolical and elliptical type in a natural way are used to solve some of the initial and boundary problems of partial differential equations such as wave equations – the filters of hyperbolical type and Laplace's equations – the filters of elliptical type. However, the function filters of the real power can be applied to the synthesis of some systems with lumped parameters, such as phase shifters, synthesis of which in the analog case is very inconvenient. The integral – derivative fractional order complex operators can also be used to determine a boundary impedance of an infinite ladder circuit.

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