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Positional transformation

In this paper the new transformation of coordinates in three-dimensional space has been given. This transformation allows determination of the coordinates of the points in the new Cartesian coordinate system based on the reference point with known coordinates in initial Cartesian coordinate system.

1. *Introduction*

In some branches of science concerning problems in space research it is often found necessary to determine the position of the points relative to the points which are taken as the reference points. The reference points are the points in space which coordinates are known in the initial Cartesian coordinate system.

In this paper new transformation for determining the position of the points in relation to the reference points of known coordinates in initial Cartesian coordinate system, has been made. Numerical example demonstrates the validity of this transformation.

2. *Transformation of the positions*

Let the set of the points, which positions are known in *xyz* Cartesian coordinate system, be given. Assuming that the points A, B and C are not lying on the same straight line, let us introduce the new *x'y'z'* Cartesian coordinate system. In this system we establish the position of any point P belonging to the remaining ones.

Let the point A be the origin of the new coordinate system, the *x'*-axis passes through the points A and B, the *x'y'*-plane contains the point C, and $\boldsymbol{r}$ be the position vector of the point P, Fig. 1.

To determine the position of the point P let us introduce the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$, for which the point A is the initial and the points B and C are the terminals, respectively. The components of these vectors may be written in the form

$$\boldsymbol{b} = [b_x, b_y, b_z] \tag{1}$$

and

$$\boldsymbol{c} = [c_x, c_y, c_z] \tag{2}$$

Considering the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ we can establish the orthogonal vectors $\boldsymbol{u}$, $\boldsymbol{\upsilon}$, $\boldsymbol{w}$ beeing the axis vectors of the new coordinate system, $x'y'z'$. Denoting the vector $\boldsymbol{b}$, lying on the axis x', by $\boldsymbol{u}$, we write

$$\boldsymbol{u} = \boldsymbol{b} \tag{3}$$

and from the definition of the vector product

$$\boldsymbol{w} = \boldsymbol{b} \times \boldsymbol{c} \tag{4}$$

and

$$\boldsymbol{\upsilon} = (\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{b} \tag{5}$$

The components of the vector $\boldsymbol{u}$ (3), have the form

$$\boldsymbol{u} = [b_x, b_y, b_z] \tag{6}$$

or

$$\boldsymbol{u} = [u_x, u_y, u_z] \tag{7}$$

For the vector $\boldsymbol{w}$ (4), we get

$$\boldsymbol{w} = [(\boldsymbol{b}\times\boldsymbol{c})_x, (\boldsymbol{b}\times\boldsymbol{c})_y, (\boldsymbol{b}\times\boldsymbol{c})_z] \tag{8}$$

or

$$\boldsymbol{w} = [w_x, w_y, w_z] \tag{9}$$

Analogically, for the vector $\boldsymbol{\upsilon}$ (5), we have

$$\boldsymbol{\upsilon} = [((\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{b})_x, ((\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{b})_y, ((\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{b})_z] \tag{10}$$

or

$$\boldsymbol{\upsilon} = [\upsilon_x, \upsilon_y, \upsilon_z] \tag{11}$$

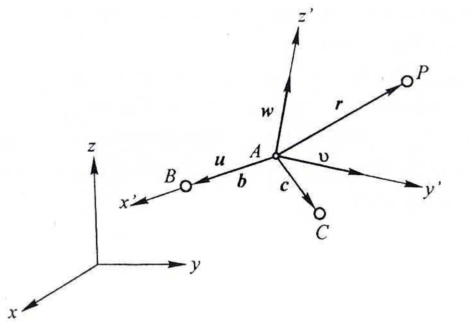

Fig. 1. Coordinate system in space

The obtained components of vectors, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, after introducing the unit vectors of xyz coordinate system denoted by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, can be written as follows

$$\boldsymbol{u} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k} \tag{12}$$

in determinant form

$$w = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \tag{13}$$

and

$$v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ w_x & w_y & w_z \\ b_x & b_y & b_z \end{vmatrix} \tag{14}$$

The unit vectors of the axis of the xyz coordinate system are

$$\boldsymbol{i} = [1, \quad 0, \quad 0] \tag{15}$$

$$\boldsymbol{j} = [0, \quad 1, \quad 0] \tag{16}$$

$$\boldsymbol{k} = [0, \quad 0, \quad 1] \tag{17}$$

Having established the vectors $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$ and unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, we can determine the cosines of angles between them. By definition, the cosine of the angle between any pair of nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{e}$ has the form

$$\cos(a,e) = \frac{\boldsymbol{a} \times \boldsymbol{e}}{ae} \tag{18}$$

where $\boldsymbol{a} \times \boldsymbol{e}$ is the inner product.

On the base of the above function we can determine the cosines of the angles between the unit vectors of the xyz coordinate system, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, and the axis vectors of $x'y'z'$ coordinate system, $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. Substituting (15), (16), (17) together with (12), (13), (14) into (18), and denoting the angles by the coordinate axes we obtain

$$\cos(x,x') = \frac{u_x}{u} \qquad \cos(y,x') = \frac{u_y}{u} \qquad \cos(z,x') = \frac{u_z}{u} \tag{19}$$

$$\cos(x,y') = \frac{v_x}{v} \qquad \cos(y,y') = \frac{v_y}{v} \qquad \cos(z,y') = \frac{v_z}{v} \tag{20}$$

$$\cos(x,z') = \frac{w_x}{w} \qquad \cos(y,z') = \frac{w_y}{w} \qquad \cos(z,z') = \frac{w_z}{w} \tag{21}$$

where

$$u = \sqrt{u_x^2 + u_y^2 + u_z^2} \tag{22}$$

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{23}$$

$$w = \sqrt{w_x^2 + w_y^2 + w_z^2} \tag{24}$$

are the lengths of the vectors.

If we make out the matrix of the cosines of the angles, (19), (20), (21), and take into account the position vector of the point P

$$\boldsymbol{r} = [r_x, r_y, r_z] \tag{25}$$

we can determine the coordinates which are demanded.

Introducing the matrix of the angles cosines and the position vector, the coordinates of the point P in *x'y'z'* coordinate system, we write in the form

$$\begin{bmatrix} x'_P \\ y'_P \\ z'_P \end{bmatrix} = \begin{bmatrix} \frac{u_x}{u} & \frac{u_y}{u} & \frac{u_z}{u} \\ \frac{v_x}{v} & \frac{v_y}{v} & \frac{v_z}{v} \\ \frac{w_x}{w} & \frac{w_y}{w} & \frac{w_z}{w} \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \tag{26}$$

where the matrix, in terms of the components of vectors, is the orthogonal matrix.

For simplicity, if we denote the elements of the matrix as follows

$$u_x^0 = \frac{u_x}{u} \qquad u_y^0 = \frac{u_y}{u} \qquad u_z^0 = \frac{u_z}{u} \tag{27}$$

$$v_x^0 = \frac{v_x}{v} \qquad v_y^0 = \frac{v_y}{v} \qquad v_z^0 = \frac{v_z}{v} \tag{28}$$

$$w_x^0 = \frac{w_x}{w} \qquad w_y^0 = \frac{w_y}{w} \qquad w_z^0 = \frac{w_z}{w} \tag{29}$$

we can write (26) in the form

$$\begin{bmatrix} x'_P \\ y'_P \\ z'_P \end{bmatrix} = \begin{bmatrix} u_x^0 & u_y^0 & u_z^0 \\ v_x^0 & v_y^0 & v_z^0 \\ w_x^0 & w_y^0 & w_z^0 \end{bmatrix} \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} \tag{30}$$

what determines the position of the point P in *x'y'z'* coordinate system.

3. Numerical example

In *xyz* coordinate system four points 1, 2, 3 and 4 have the following coordinates 1(1 , 1 , 1), 2(2 , 2 , 3), 3(2 , 2 , 2), 4(0 , 2 , 2). Find coordinates of the point 4 in new *x'y'z'* coordinate system the origin of which is at the point 1, the *x'*-axis is passing through the point 2, and the point 3 is lying on the *x'y'*-plane.

Solution

The vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ have the components

$$\boldsymbol{b} = [1 \, , 1 \, , 2]$$
$$\boldsymbol{c} = [1 \, , 1 \, , 1]$$

so that

$$\boldsymbol{w} = \boldsymbol{b} \times \boldsymbol{c} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{vmatrix} = [-1, 1, 0]$$

and

$$\upsilon = (\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = [2, 2, -2]$$

Then, we obtain

$$\begin{bmatrix} x'_4 \\ y'_4 \\ z'_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{2}} \end{bmatrix}$$

where x'_4 , y'_4 , z'_4 are the coordinates of the point 4 in *x'y'z'* coordinate system.

Check

To verify the obtained result we calculate the distance between the point 3 and 4 in both coordinate systems. Determining the coordinates of the point 3 in the *x'y'z'* coordinate system, we have

$$\begin{bmatrix} x_3' \\ y_3' \\ z_3' \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}$$

Note that the point 3 is realy lying on the $x'y'$-plane because $z_3' = 0$.

The distance between the points 3 sand 4 in the $x'y'z'$ coordinate system

$$d_{3,4} = \sqrt{\left(\frac{2}{\sqrt{6}} - \frac{4}{\sqrt{6}}\right)^2 + \left(-\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)^2 + \left(\frac{2}{\sqrt{2}} - 0\right)^2} = 2$$

and in xyz coordinate system

$$d_{3,4} = \sqrt{(0-2)^2 + (2-2)^2 + (2-2)^2} = 2$$

then $d_{3,4}' = d_{3,4}$ what should be expected.

4. *Final remarks*

Owing to the obtained transformation we can determine the position of the points relative to the points in space which are taken as the reference points. This situation appears in space navigation, and in determination of positions by satellite techniques.

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Transformacja pozycyjna

S t r e s z c z e n i e

W pracy podano nową transformację współrzędnych w przestrzeni trójwymiarowej, którą nazwano transformacją pozycyjną.

Transformacja ta, w odróżnieniu od znanych transformacji ustalanych w wyniku obrotów i translacji układu, zapewnia bezpośrednie wyznaczanie pozycji w układzie wtórnym, na podstawie znanych współrzędnych w układzie pierwotnym. Otrzymana transformacja pozwala na bezpośrednie rozwiązywanie podstawowych problemów geodezji przestrzennej i nawigacji satelitarnej.

Януш Мартусевич

Позиционная трансформация

Р е з ю м е

В работе представлена новая трансформация координат в трёхмерном пространстве, которая названа позиционной трапсформацией.

Эта трансформация, в отличии от известных других трансформаций проводимых в результате оборотов и трансляции системы, обеспечивает прямое определение позиции во вторичной системе на основе известных координат в первоначальной системе. Получена трансформация даёт возможность прямого решения основных проблем пространственной геодезии и спутниковой навигации.