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Mean error of a function of intermediary unknowns and observations which are not the subject of adjustment in the system

In the paper the formula for equation of the observation correction is introduced, which also includes observation terms, which are not the subjects of adjustment. The system of such type of correction equations is the basis for calculation of intermediary unknowns, which are not only the function of observations being the subjects of adjustment, but also of observations, which are not deformed by corrections. The paper presents proofs of formulae for calculation of mean errors of intermediary unknowns and functions of those unknowns for a considered case. An important general conclusion results from those proofs: both, the mean error of the *i*th unknown, and the mean error of the function of unknown, obtained from the discussed system of equations can not be smaller than the corresponding error obtained form a system, which does not include those observations.

Presented formulae may be used in the case of adjustment of a connected network to higher order points, which co-ordinates are considered as observations, which are not the subjects of adjustment. Therefore we assume them as constant in a narrower range, i.e. we assume constancy of their values after adjustment; however their mean errors are considered in accuracy analysis.

Derived formulae may be also used for calculation of mean errors of explicitly determinable geodetic constructions connected to known points, if the influence of mean errors of co-ordinates of those points on the mean error of a given function of unknowns is to be considered.

Let us assume that adjustment of n observations L^{ob} is performed in order to determine s unknown X, with the use of a procedure of the intermediary method. However, besides unknowns L^{ob}, X, observation equations contains also r observations C^{ob}, which are not the subject of adjustment. Having mean errors m of observations L^{ob}, which are to be adjusted and mean errors M of observations C^{ob}, which are not the subject of adjustment, we will intend to derive formulae for the mean error of the ith unknown X_i and the function $E = E(X, C^{ob})$ of unknowns X and observations C^{ob}, which are not the subject of adjustment. Therefore, the idea is to consider the mean errors M of observations C^{ob} in accuracy analysis after adjustment of the system.

Let us consider the ith observation equation of the discussed type:

$$L_i^w = F_i(X_1, X_2, \ldots, X_z, C_1^{ob}, C_2^{ob}, \ldots, C_r^{ob}) \tag{1}$$

where:

$$i = 1,2, \dots , n \tag{2}$$

and L_i^w is the *i*th adjusted observation.

Let us assume that we know approximate values of X_{j0}, C_{q0}' ($j = 1,2, \dots ,s$; $q = 1,2, \dots ,r$) of unknowns X and observations C, which are differ from the proper values by x_j , c_q, i.e.:

$$X_j = X_{j0} + x_j \qquad ; \qquad C_q = C_{q0} + c_q \tag{3}$$

For the initial values X_{j0}, C_{q0} we will develop the function F_i into the Taylor series:

$$\begin{aligned} L_i^w = F_i(X_{10}, X_{20}, \dots, X_{z0}, C_{10}, C_{20}, \dots, C_{r0},) + \frac{\partial F_i}{\partial X_1} x_1 + \frac{\partial F_i}{\partial X_2} x_2 + \dots + \frac{\partial F_i}{\partial X_s} x_s + \\ + \frac{\partial F_i}{\partial C_1} c_1 + \frac{\partial F_i}{\partial C_2} c_1 + \dots + \frac{\partial F_i}{\partial C_r} c_r \end{aligned} \tag{4}$$

We will introduce the symbols:

$$L_i^{\mathrm{prz}} = F_i(X_{10}, X_{20}, \dots, X_{s0}, C_{10}, C_{20}, \dots, C_{r0})$$

$$\frac{\partial F_i}{\partial X_1} = a_i, \frac{\partial F_i}{\partial X_2} = b_i, \dots, \frac{\partial F_i}{\partial X_s} = s_i, \frac{\partial F_i}{\partial C_1} = \bar{i}_1, \frac{\partial F_i}{\partial C_2} = \bar{i}_2, \dots, \frac{\partial F_i}{\partial C_r} = \bar{i}_r \tag{5}$$

Remembering that the adjusted observation is the sum of the observation and the observation correction

$$L_i^w = L_i^{\mathrm{ob}} + \upsilon_i \tag{6}$$

and having

$$l_i = L_i^{\mathrm{prz}} - L_i^{\mathrm{ob}} \tag{7}$$

basing on (4), we will obtain the equation of the *i*th correction in the form:

$$\upsilon_i = a_i x_1 + b_i x_2 + \dots + s_i x_s + \bar{i}_1 c_1 + \bar{i}_2 c_2 + \dots + \bar{i}_r c_r + l_i \tag{8}$$

which free term

$$u_i = \bar{i}_1 c_1 + \bar{i}_2 c_2 + \dots + \bar{i}_r c_r + l_i \tag{9}$$

is the function of r terms of observation terms c_q, which have the mean errors M_q, and of one observation term l_i of he mean error m_i.

We will equalise this equation in the following way: First of all, components of its both sides will be divided by the mean error of the ith observation m_i, and then every transformed coefficient, which occurs with the variable c_q will be multiplied by M_q, c_q will be divided by M_q. Thus, the following equation will be obtained:

$$\frac{\upsilon_i}{m_i}=\frac{a_i}{m_i}x_1+\frac{b_i}{m_i}x_2+\ldots+\frac{s_i}{m_i}x_s+\frac{\bar{i}_1}{m_i}M_1\frac{c_1}{M_1}+\frac{\bar{i}_2}{m_i}M_2\frac{c_2}{M_2}+\ldots+\frac{\bar{i}_r}{m_i}M_r\frac{c_r}{M_r}+\frac{l_i}{m_i} \quad (10)$$

The following symbols will be introduced:

$$V_i=\frac{\upsilon_i}{m_i};\ A_i=\frac{a_i}{m_i};\ B_i=\frac{b_i}{m_i};\ldots;\ S_i=\frac{s_i}{m_i};\ L_i=\frac{l_i}{m_i};$$
$$\overline{I}_1=\frac{\bar{i}_1M_1}{m_i};\ \overline{I}_2=\frac{\bar{i}_2M_2}{m_i};\ldots,\ \overline{I}_r=\frac{\bar{i}_rM_r}{m_i};\ C_q=\frac{c_q}{M_q}; \quad (11)$$

which allow to write the equalised, ith correction equation in the form:

$$V_i=A_ix_1+B_ix_2+\ldots+S_ix_s+\overline{I}_1C_1+\overline{I}_2C_2+\overline{\mathrm{I}}_rC_r+L_i \quad (12)$$

and, similarly, n such equations:

$$V_1=A_1x_1+B_1x_2+\ldots+S_1x_s+\overline{A}_1C_1+\overline{A}_2C_2+\overline{A}_rC_r+L_1$$
$$V_2=A_2x_1+B_2x_2+\ldots+S_2x_s+\overline{B}_1C_1+\overline{B}_2C_2+\ldots+\overline{B}_rC_r+L_2 \quad (13)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$V_n=A_2x_1+B_2x_2+\ldots.+S_2x_s+\overline{B}_1C_1+\overline{B}_2C_2+\ldots.+\overline{B}_rC_r+L_2$$

Marking of the following cracovians:

$$\mathbf{V}=\begin{Bmatrix}V_1\\V_2\\\ldots.\\V_n\end{Bmatrix};\mathbf{x}=\begin{Bmatrix}x_1\\x_2\\\ldots.\\x_s\end{Bmatrix};\mathbf{A}=\begin{Bmatrix}A_1B_1\ldots S_1\\A_2B_2\ldots S_2\\\ldots\ldots\ldots\ldots\\A_nB_n\ldots S_n\end{Bmatrix};\mathbf{L}=\begin{Bmatrix}L_1\\L_2\\\ldots.\\L_n\end{Bmatrix};\mathbf{C}=\begin{Bmatrix}C_1\\C_2\\\ldots.\\C_r\end{Bmatrix};\mathbf{B}=\begin{Bmatrix}\overline{A_1A_2\ldots A_r}\\\overline{B_1B_2\ldots B_r}\\\ldots\ldots\ldots\ldots\\\overline{N_1N_2\ldots N_r}\end{Bmatrix} \quad (14)$$

allow to present the equalised system of correction equation in the following way:

$$\mathbf{V}=\mathbf{x}\tau\mathbf{A}+\mathbf{C}\tau\mathbf{B}+\mathbf{L} \quad (15)$$

where the free term **U** of the above cracovian equation

$$\mathbf{U} = \mathbf{C}\tau\mathbf{B} + \mathbf{L} \tag{16}$$

is a block of *n* rows and *r*+1 columns. Therefore we may simplify the notation of the relation:

$$\mathbf{V} = \mathbf{x}\tau\mathbf{A} + \mathbf{U} \tag{17}$$

The normal equations, based on (17) have the form

$$\mathbf{x}\mathbf{A}^2 + \mathbf{U}\mathbf{A} = \mathbf{0} \tag{18}$$

and they have the solution

$$\mathbf{x} = -\mathbf{U}\mathbf{A}\left(\mathbf{A}^2\right)^{-1}$$

or, after association

$$\mathrm{x} = -\mathrm{U}\left\{(\mathrm{A}^2)^{-1}\tau\mathrm{A}\right\} \tag{19}$$

Remembering the known notation

$$\mathbf{T} = \left(\mathbf{A}^2\right)^{-1}\tau\mathbf{A} \tag{20}$$

we have

$$\mathbf{x} = -\mathbf{U}\mathbf{T} \tag{21}$$

i.e., after taking (16) into account

$$\mathrm{x} = -\left(\mathrm{C}\tau\mathrm{B} + \mathrm{L}\right)\mathrm{T} = -\mathrm{C}\tau\mathrm{B}\mathrm{T} - \mathbf{L}\mathbf{T}$$

or

$$\mathbf{x} = -\mathbf{C}\left(\mathbf{T}\mathbf{B}\right) - \mathbf{L}\mathbf{T} \tag{22}$$

The above formula presents unknowns **x** as functions of equalised observation terms **C, L.** Since

$$x_i = -\mathbf{C}(\mathbf{TB})_i - \mathbf{LT}_i$$

the *i*th unknown is obtained by multiplication the observation cracovian **C** by the *i*th column of the product **TB** and by adding the product of the observation cracovian **L** by the *i*th column of the cracovian **T**; so we can write the formula for the mean error of the *i*th unknown:

$$m_{xi}^2 = m_0^2\left\{[(\mathrm{TB})_i]^2 + [(\mathrm{T})_i]^2\right\} \tag{23}$$

It may be simply justified, that

$$\mathbf{T}^2 = (\mathbf{A}^2)^{-1}$$

so:

$$(\mathbf{T}_i)^2 = [(\mathbf{A}^2)^{-1}]_{ii} \tag{24}$$

therefore we can write

$$m_{xi}^2 = m_0^2\{[(\mathrm{TB})_i]^2 + [(\mathrm{A}^2)^{-1}]_{ii} \tag{25}$$

If observations, which are not the subject of adjustment, do not occur in correction equations, the first term disappears and the known relation is obtained:

$$(m_{xi}^2)' = m_0^2[(\mathbf{A}^2)^{-1}]_{ii} \tag{26}$$

It turns out that the mean error of the *i*th unknown obtained from adjustment of the system, where observations, which are not the subject of adjustment appear, cannot be smaller than the mean error obtained from the system in which such observations do not occur.

The formula (26) will be used when the coefficient m_0 is obtained from equalised corrections

$$m_0 = \sqrt{\frac{VV}{n-s}} \tag{27}$$

However, in the case of initial accuracy analysis, when exact values of mean errors assumed for equalising are considered, the following formula will be used

$$m_{xi} = [(\mathbf{TB})_i]^2 + [(\mathbf{A}^2)^{-1}]_{ii} \tag{28}$$

since $m_0 = 1$ in this case.

Let us now assume that the following function is given

$$E = E(X_1, X_2, ..., X_s, C_1^{ob}, C_2^{ob}, ..., C_r^{ob}) \tag{29}$$

which is the function of intermediary unknowns $\mathbf{X}$ and observations $\mathbf{C}^{ob}$ which are not the subject of adjustment. For already assumed initial values X_{j0}, C_{q0} and after considering of relations (3) it will be developed into Taylor's series

$$\begin{aligned} E = E(X_{10}, X_{20}, ..., X_{s0}, C_{10}, C_{20}, ..., C_{r0}) + \frac{\partial E}{\partial X_1} x_1 + \frac{\partial E}{\partial X_2} x_2 + ... + \frac{\partial E}{\partial X_s} x_s + \\ + \frac{\partial E}{\partial C_1^{ob}} c_1 + \frac{\partial E}{\partial C_2^{ob}} c_2 + ... + \frac{\partial E}{\partial C_r^{ob}} c_r \end{aligned} \tag{30}$$

The following notations will be introduced

$$E_0 = E(X_{10}, X_{20}, ..., X_{s0}, C_{10}, C_{20}, ..., C_{r0}); \frac{\partial E}{\partial X_j} = f_j; \frac{\partial E}{\partial C_q^{ob}} = g_q \tag{31}$$

which will allow to write (30) in the following form

$$E = f_1 x_1 + f_2 x_2 + ... + f_s x_s + g_1 c_1 + g_2 c_2 + ... + g_r c_r + E_0 \tag{32}$$

E_0 is the constant, which does not influence the error of the function E. We will equalise the function E, writing it in the form, which does not contain E_0

$$E = f_1 x_1 + f_2 x_2 + ... + f_s x_s + g_1 M_1 \frac{c_1}{M_1} + g_2 M_2 \frac{c_2}{M_2} .... + g_r M_r \frac{c_r}{M_r} \tag{33}$$

Considering the assumed notation

$$C_q = \frac{c_q}{M_q} \tag{34}$$

and introducing the new notation

$$G_q = g_q M_q \tag{35}$$

we will obtain the following expression for calculation of E

$$E = f_1 x_1 + f_2 x_2 + ... + f_s x_s + G_1 C_1 + G_2 C_2 + ... + G_r C_r \tag{36}$$

Notation of function cracovians

$$\tau f = \{f_1 f_2 ... f_s\}; \tau G = \{G_1 G_2 ... G_r\} \tag{37}$$

enables the cracovian notation of the function E

$$E = \mathbf{xf} + \mathbf{CG} \tag{38}$$

After substitution of (22) which expresses **x** as the function of **C, L** we will obtain

$$E = -\{\mathbf{C(TB)} - \mathbf{LT}\}\mathbf{f} + \mathbf{CG} = -\mathbf{C(TB)f} - \mathbf{LTf} + \mathbf{CG} = \mathbf{C}\{\mathbf{G} - \mathbf{f(BT)}\} - \mathbf{L(f\tau T)}$$

So, we have

$$m_E^2 = m_0^2\{(\mathbf{f\tau T})^2 + [\mathbf{G} - \mathbf{f(BT)}]^2\}$$

and knowing, that

$$(\mathbf{f\tau T}^2 = \mathbf{f(A}^2)^{-1}\mathbf{f}$$

we obtain the following formula for calculating m_E

$$m_E^2 = m_0^2\{\mathbf{f}(\mathbf{A}^2)^{-1}\mathbf{f} + [\mathbf{G} - \mathbf{f(BT)}]^2\} \tag{39}$$

Obviously, in the case of initial accuracy analysis we will use the following formula

$$m_E^2 = \{\mathbf{f}(\mathbf{A}^2)^{-1}\mathbf{f} + [\mathbf{G} - \mathbf{f(BT)}]^2\} \tag{40}$$

We should note that when observations, which are not the subject of adjustment do not occur in the system, $\mathbf{G} = \mathbf{0}$ and $\mathbf{B} = \mathbf{0}$; from (39) the known formula for the mean error of the function of intermediary unknowns is obtained

$$m_E^2 = \mathbf{f}(\mathbf{A}^2)^{-1}\mathbf{f} \tag{41}$$

It turns out from here, that the mean error of unknowns and observations, which are not being adjusted, cannot be smaller than the mean error of unknowns.

The above procedure can be also applied for systems, which are explicitly determined, if from the set of observations $s + r$, s observations may be selected, for which it is possible to arrange s observation equations of (1) type, out of which the ith equation presents the observation L_i (one observation) as the function of X, C, i.e. $L_i^{ob} = F_i\,(X,C)$.

Although the issue apparently remains in the sphere of abstraction, developed formulae may be used for analysis of accuracy of the network connected to points of the higher order, after adjustment performed with the use of the intermediary method, as well as for determination of mean errors of functions of unknowns in constructions, which can be explicitly determined, basing on given points. If, keeping unchanged values of co-ordinates of such points, we would like to consider their influence on interesting functions of variables, co-ordinates of tie points should be assumed as observations, which are not the subject of adjustment. The algorithm is slightly complicated, however it may become significantly useful in some cases.

In practice, the problem relies upon arrangement of correction equations of real observations, with the consideration of co-ordinates of tie points as variables and with such sequence of unknowns in the system of correction equation, that unknowns related to points being determined and to tie points occur separately. After equalisation of the system according to the described procedure, the normal equations are arranged basing on **A, L** blocks and the block **B**, which concerns "non-adjusted" observations, i.e. co-ordinates of tie points, is used only in the course of calculation of mean errors of given functions. It is obviously correct, when, in the development into Taylor's series, co-ordinates of given points are assumed as equal to catalogue values, what usually happens. The procedure is illustrated by the following examples.

E x a m p l e 1. 3 levelling lines were measured between points 0 and 3 and the following values of height differences and corresponding mean errors were obtained:

$h_1 = 1001$ m $\qquad m_1 = 0.5$ mm
$h_2 = 1998$ m $\qquad m_2 = 2$ mm
$h_3 = 2999$ m $\qquad m_3 = 0.5$ mm

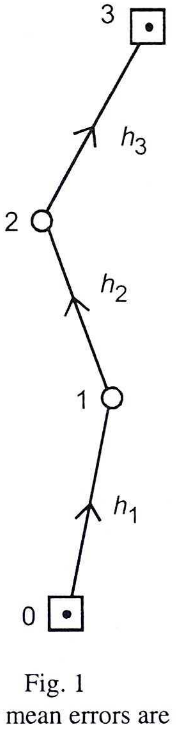

Fig. 1

Levels of tie benchmarks 0 and 3 and their mean errors are equal to

$Z_0 = 10000$ mm $\qquad M_0 = 2$ mm
$Z_3 = 16000$ mm $\qquad M_3 = 0.5$ mm

To adjust this system by means of the intermediary method, with the assumption of stability of tie points and to calculate mean errors of unknown heights Z_1 and Z_2, considering errors of levels of benchmarks Z_0, Z_1 and the mean error of the difference in height $\Delta = Z_2 - Z_3$. Approximate heights of points are equal to

$$Z_{10} = 11000 \text{ mm}$$
$$Z_{20} = 13000 \text{ mm}$$

We assume that levels of tie benchmarks are observations of known mean errors, which are not the subject of adjustment. Approximate values of those observations are assumed as equal to catalogue values, i.e. to given values, and correction equations will be arranged considering height intervals of tie benchmarks. The initial, non-adjusted system will have the form

$$\upsilon_1 = 1z_1 + 0z_2 - 1z_0 + 0z_3 + 1$$
$$\upsilon_2 = -1z_1 + 0z_2 + 0z_0 + 0z_3 + 2$$
$$\upsilon_3 = 0z_1 - 1z_2 + 0z_0 + 1z_3 - 1$$

This system will be written in the simplified form

i	z_1	z_2	z_0	z_3	1
1	1	0	–1	0	1
2	–1	1	0	0	2
3	0	–1	0	1	–1

Since approximate values of non-adjusted observations are equal to "observed", i.e. catalogue values, so intervals z_0, z_3 are equal to zero, therefore two initial columns, assigned to unknowns z_1, z_2 and the column of free terms of the following equalised system will be used for specification of the normal equations:

i	z_1	z_2	z_0	z_3	1
1	2	0	–4	0	2
2	–0.5	0.5	0	0	1
3	0	–2	0	1	–2

in which the following blocks may be distinguished:

$$\mathbf{A} = \left[\frac{1}{2} \begin{Bmatrix} 4 & 0 \\ -1 & 1 \\ 0 & -4 \end{Bmatrix} \right]; \ \mathbf{B} = \begin{Bmatrix} -4 & 0 \\ 0 & 0 \\ 0 & 1 \end{Bmatrix}; \ \mathbf{L} = \begin{Bmatrix} 2 \\ 1 \\ -2 \end{Bmatrix}$$

They enable to solve the problem, since

$$\mathbf{A}^2=\left[\frac{1}{2}\begin{Bmatrix}4 & 0\\ -1 & 1\\ 0 & -4\end{Bmatrix}\right]=\frac{1}{4}\begin{Bmatrix}17 & -1\\ -1 & 17\end{Bmatrix};\ (\mathbf{A}^2)^{-1}=\frac{1}{72}\begin{Bmatrix}17 & 1\\ 1 & 17\end{Bmatrix};\ \mathbf{LA}=\frac{1}{2}\begin{Bmatrix}2\\ 1\\ -2\end{Bmatrix}\begin{Bmatrix}4 & 0\\ -1 & 1\\ 0 & -4\end{Bmatrix}=\frac{1}{2}\begin{Bmatrix}7\\ 9\end{Bmatrix}$$

thus the unknowns may be easily obtained

$$\begin{Bmatrix}z_1\\ z\end{Bmatrix}=-\frac{1}{144}\begin{Bmatrix}7\\ 9\end{Bmatrix}\begin{Bmatrix}17 & 1\\ 1 & 17\end{Bmatrix}=-\frac{1}{144}\begin{Bmatrix}128\\ 160\end{Bmatrix}=\begin{Bmatrix}-0.8889\\ -1.1111\end{Bmatrix}$$

not leaving the known algorithm of the intermediary method. However, mean errors of unknowns Z_1, Z_3 will be calculated in an untypical way. Calculations should be based on cracovians **T** and **TB**.

$$\mathbf{T}=(\mathbf{A}^2)^{-1}\tau\mathbf{A}=\frac{1}{72}\begin{Bmatrix}17 & 1\\ 1 & 17\end{Bmatrix}\frac{1}{2}\begin{Bmatrix}4 & -1 & 0\\ 0 & 1 & -4\end{Bmatrix}=\frac{1}{144}\begin{Bmatrix}68 & 4\\ -16 & 16\\ -4 & -68\end{Bmatrix}=\frac{1}{36}\begin{Bmatrix}17 & 1\\ -4 & 4\\ -1 & -17\end{Bmatrix}$$

$$\mathbf{TB}=\frac{1}{36}\begin{Bmatrix}17 & 1\\ -4 & 4\\ -1 & -17\end{Bmatrix}\begin{Bmatrix}-4 & 0\\ 0 & 0\\ 0 & 1\end{Bmatrix}=\frac{1}{36}\begin{Bmatrix}-68 & -4\\ -1 & -17\end{Bmatrix}$$

which allow to calculate mean errors of unknowns

$$m_{z1}^2=m_0^2\left\{\frac{1}{36^2}\begin{Bmatrix}-68\\ -1\end{Bmatrix}^2+\frac{17}{72}\right\}=m_0^2(3.5687+0.2361)=3.8048m_0^2$$

$$m_{z2}^2=m_0^2\left\{\frac{1}{36^2}\begin{Bmatrix}-4\\ -17\end{Bmatrix}^2+\frac{17}{72}\right\}=m_0^2(0.2353+0.2361)=0.4614m_0^2$$

Since equalised corrections have the values

$$V_1=0.2222;\quad V_2=0.8889;\quad V_3=0.2222$$

we can calculate the coefficient m_0

$$m_0=\sqrt{\frac{VV}{3-1}}=\sqrt{0.8889}=0.9428$$

and the values of mean errors of unknowns

$$m_{z1} = 1.8390\ ; m_{z2} = 0.6473$$

may be easily calculated.

Calculation of the mean error of the function Δ requires specification of function cracovians. Since

$$\Delta = Z_2 \quad - Z_1$$

so

$$\frac{\partial \Delta}{\partial Z_1} = 0; \frac{\Delta}{\partial Z_2} = 1; \rightarrow \tau f = \{0\ 1\}$$

$$\frac{\partial \Delta}{Z_0} = -1; \frac{\partial \Delta}{\partial Z} = 0; \rightarrow \tau \mathbf{g} = \{-1 \quad 0\}$$

we have to equalise the cracovian **g**

$$\tau \mathbf{G} = \{-2 \quad 0\}$$

Now the components of the formula (39) may be calculated

$$\mathbf{f(BT)} = \frac{1}{36}\begin{Bmatrix}0\\1\end{Bmatrix}\begin{Bmatrix}-68 & -1\\-4 & -17\end{Bmatrix} = \frac{1}{36}\begin{Bmatrix}-4\\-17\end{Bmatrix}$$

$$\mathbf{G - f(BT)} = \frac{1}{36}\begin{Bmatrix}-72\\0\end{Bmatrix} - \frac{1}{36}\begin{Bmatrix}-4\\-17\end{Bmatrix} = \frac{1}{36}\begin{Bmatrix}-68\\17\end{Bmatrix}$$

$$\{\mathbf{G - f(BT)}\}^2 = 3.7909$$

$$\mathbf{f}(\mathbf{A}^2)^{-1}\mathbf{f} = \frac{1}{72}\begin{Bmatrix}0\\1\end{Bmatrix}\begin{Bmatrix}17 & 1\\1 & 17\end{Bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix} = \frac{1}{72}\begin{Bmatrix}1\\17\end{Bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix} = \frac{17}{72} = 0.2361$$

After substitutions we will obtain

$$m_\Delta^2 = m_0^2(3.7909 + 0.2361) = 1.8920$$

The advantage of this procedure is maintenance the standard dimension sxs of the table $\mathbf{A}^2$ and the scope of usefulness of the algorithm, which allows to use it for the analysis of accuracy of adjusted systems as well as of systems, which are explicitly determined. It disadvantage is related too complicated equalisation and the necessity to calculate transforming cracovians.

E x a m p l e 2. In a given open polygon the mean error of the tie benchmark no 0 is equal to M_0 = 2 mm and mean errors of observations, h_1, h_2 are equal to m_1= 0.5 mm and m_2 = 2 mm, respectively. To calculate the mean errors of the following functions:
1) of the Z_1, Z_2 heights of the benchmarks no 1 and 2,
2) of the difference in height $\Delta = Z_2 - Z_0$ of the benchmarks no 2 and 0,
3) of the difference in height $d = Z_2 - Z_1$ of the benchmarks no 2 and 1,
considering the influence of the error of he tie benchmark.

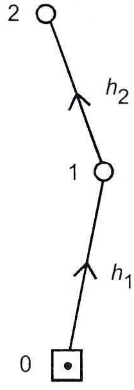

Fig. 2

The construction may be explicitly determined and the problem aims at calculation of mean errors of the mentioned functions. So we arrange equations of intervals of observations h_i, making the height of the tie benchmark variable

i	z_1	z_2	z_0	m_i
1	1	0	–1	0.5
2	–1	1	0	2
			2	M_q

The system will be equalised and we will obtain after equalisation

i	z_1	z_2	z_0	m_i
1	2	0	-4	0.5
2	–0.5	0.5	0	2
			2	M_q

The following blocks will be distinguished in this system:

$$\mathbf{A} = \frac{1}{2}\begin{Bmatrix} 4 & 0 \\ -1 & 1 \end{Bmatrix}; \mathbf{B} = \begin{Bmatrix} -4 \\ 0 \end{Bmatrix}$$

and we will obtain

$$\mathbf{A}^2 = \frac{1}{4}\begin{Bmatrix} 17 & -1 \\ -1 & 1 \end{Bmatrix}; (\mathbf{A}^2)^{-1} = \frac{1}{4}\begin{Bmatrix} 1 & 1 \\ 1 & 17 \end{Bmatrix}$$

$$\mathbf{T} = (\mathbf{A}^2)^{-1}\tau\mathbf{A} = \frac{1}{4}\begin{Bmatrix} 1 & 1 \\ 1 & 17 \end{Bmatrix}\frac{1}{2}\begin{Bmatrix} 4 & -1 \\ 0 & 1 \end{Bmatrix} = \frac{1}{8}\begin{Bmatrix} 4 & 4 \\ 0 & 16 \end{Bmatrix} = \frac{1}{2}\begin{Bmatrix} 1 & 1 \\ 0 & 4 \end{Bmatrix}$$

$$\mathbf{TB} = \frac{1}{2}\begin{Bmatrix} 1 & 1 \\ 0 & 4 \end{Bmatrix}\begin{Bmatrix} -4 \\ 0 \end{Bmatrix} = -2\{1 \ \ 1\}$$

So, according to (28) we obtain

$$m_{z1}^2 = [(\mathbf{TB})_1]^2 + [(\mathbf{A}^2)^{-1}]_{11} = (-2)^2 + 1/4 = 17/4; \rightarrow m_{z1} = \sqrt{17}/2$$

$$m_{z2}^2 = [(\mathbf{TB})_2]^2 + [(\mathbf{A}^2)^{-1}]_{22} = (-2)^2 + 17/4 = 33/4; \rightarrow m_{z2} = \sqrt{33}/2$$

Calculation of the mean error of the function

$$\Delta = Z_2 - Z_1$$

requires calculation of function cracovians **f,g**

$$\frac{\partial \Delta}{\partial Z_1} = 0; \frac{\partial \Delta}{\partial Z_2} = 1; \ \ \mathbf{f} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$\frac{\partial \Delta}{\partial Z_0} = -1; \ \ \mathbf{g} = \{-1\}$$

Cracovian **g** is equalised by multiplication of its only element by the mean error of the variable Z_0

$$\mathbf{G}=\{-2\}$$

$$\mathbf{G}-\mathbf{f(BT)}=\{-2\}-\begin{Bmatrix}0\\1\end{Bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}\{-2\}=\{-2\}+\{2\}=\{0\}$$

$$\mathbf{f}(\mathbf{A}^2)^{-1}\mathbf{f}=\begin{Bmatrix}0\\1\end{Bmatrix}\frac{1}{4}\begin{Bmatrix}1 & 1\\1 & 17\end{Bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix}=\frac{1}{4}\begin{Bmatrix}1\\17\end{Bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix}=\frac{17}{4}$$

$$m_\Delta^2=17/4 \quad m_\Delta=\sqrt{17}/2$$

Similarly the mean error of the function

$$d=Z_2-Z_1$$

is calculated

$$\frac{\partial d}{\partial Z_1}=-1;\ \frac{\partial d}{\partial Z_2}=1;\ \mathbf{f}=\begin{Bmatrix}-1\\1\end{Bmatrix};\ \frac{\partial d}{\partial Z_0}=0;\ g=\{0\};\ G=\{0\}$$

so, as it may be seen, the cracovian **G** disappears and then

$$m_d^2=\{-\mathbf{f(BT)}\}^2+\mathbf{f}(\mathbf{A}^2)^{-1}\mathbf{f}=-2\begin{Bmatrix}-1\\1\end{Bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}+\frac{1}{4}\begin{Bmatrix}-1\\1\end{Bmatrix}\begin{Bmatrix}1 & 1\\1 & 17\end{Bmatrix}\begin{Bmatrix}-1\\1\end{Bmatrix}$$

so

$$m_d^2=\frac{1}{4}\begin{Bmatrix}0\\16\end{Bmatrix}\begin{Bmatrix}-1\\1\end{Bmatrix}=4;\ m_d=2$$

In the case of the discussed, open levelling polygon, calculations may be controlled with the direct use of Gauss law. Then we will obtain

$$Z_1=Z_0+h_1;\ m_{Z1}^2=M_0^2+m_1^2=4+1/4=17/4;\ m_{Z1}=\sqrt{17}/2$$

$$Z_2=Z_0+h_1+h_2;\ m_{Z2}^2=M_0^2+m_1^2+m_2^2=4+1/4+4=33/4;\ m_{Z2}=\sqrt{33}/2$$

$$\Delta = Z_2 - Z_0 = h_1 + h_2;\ m_\Delta^2 = m_1^2 + m_2^2 = 1/4 + 4 = 17/4;\ m_\Delta = \sqrt{17}/2$$

$$d = Z_2 - Z_1 = h_2;\ m_d = m_2 = 2$$

Results (in millimetres!) are identical, of course, with results obtained earlier.

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Błąd średni funkcji niewiadomych pośredniczących i obserwacji nie podlegających wyrównaniu w układzie

S t r e s z c z e n i e

W artykule wyprowadzono wzór na równanie poprawki obserwacji, w którym występują również wyrazy obserwacyjne spostrzeżeń nie podlegających wyrównaniu. Układ tego typu równań poprawek stanowi podstawę obliczenia niewiadomych pośredniczących, które są nie tylko funkcjami obserwacji podlegających wyrównaniu, lecz także obserwacji, których nie zniekształcono poprawkami. Praca zawiera dowody formuł na obliczenie błędów średnich niewiadomych pośredniczących i funkcji tych niewiadomych dla rozważanego przypadku. Wynika z nich ważny wniosek ogólny: zarówno błąd średni *i*–tej niewiadomej, jak i funkcji niewiadomych uzyskanych z omawianego typu równań nie może być mniejszy od odnośnego błędu uzyskanego z układu, w którym te obserwacje nie występują.

Podane wzory można wykorzystać w przypadku wyrównanie sieci dowiązanych do punktów wyższej klasy, których współrzędne traktujemy jako obserwacje nie podlegające wyrównaniu. Przyjmujemy je więc za stałe w węższym zakresie tj. zakładamy niezmienność ich wartości po wyrównaniu, jednak ich błędy średnie uwzględniamy w analizie dokładności.

Wyprowadzone formuły można również wykorzystać do obliczania błędów średnich jednoznacznie wyznaczanych konstrukcji geodezyjnych dowiązanych do punktów znanych, jeśli się chce uwzględnić wpływ błędów średnich ich współrzędnych na błąd średni danej funkcji niewiadomych.

Александер Скурчыньски

Средняя квадратическая ошибка функций промежуточных неизвестных и наблюдений не подвергающих выравниванию в систиеме

Резюме

В статье выведена формула уравниения поправки наблюдений, в которой присутствуют тоже наблюдательские члены не подвергающие уравнению. Система того типа уравнений поправок является основой для вычисления промежуточных неизвестных, которые являются не только функциями наблюдений подвергающих уравнению, но тоже наблюдений, которые не были деформированы поправками. В работе представлены доказательства формул вычисления средних квадратических ошибок промежуточных неизвестных и функций этих неизвестных для рассматриваемого случая. Из них вытекает очень важный общий вывод; как средняя квадратическая ошибка u-ой неизвестной, так и функций неизвестных, полученных из обсуждаемого типа уравнений, не может получиться меньшей чем соответствующая ошибка, полученная с системы, в которой эти наблюдения не присутствуют.

Похожие формулы могут быть использованы в случае уравнивания сети, привязанных к пунктам высшего ряда, к координатам которых подходим как к наблюдениям не подвергающим уравниению. Следовательно принимаем, что они постоянны в более узком объёме, т.е. принимаем постоянство их величин после уравнения, однако их средние квадратические ошибки учитываем в анализе точности.

Выведенные формулы могут быть тоже использованы для вычисления средних квадратических ошибок однозначно определенных геодезических конструкций, привязанных к известным пунктам, когда хочется учитывать влияние средних квадратических ошибок координат этих конструкций на среднюю квадратическую ошибку данной функции неизвестных.