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Modification of Otrebski's theorem for equivalent observation systems

The paper presents a suggestion of modification of Otrebski's theorem for some special structures of geodetic networks. The modification leads to forming up the conditional equation system with unknowns. A new parameter as a global criterion of evaluation of the quality of the networks characterized by an inhomogenous observation system, has been introduced as well.

INTRODUCTION

Parameters describing the reliability features of construction of geodetic networks are mainly based on the size of set of observations and that of unknowns (see: [1, 4, 5, 6]), but they are usually correlated with their accuracy factors. Such correlation occurs in the case of networks that are characterized by a homogeneity in a sense of both the internal geometry and the system of control points. For the sake of simplicity of getting these parameters, they are often used as the criteria in accuracy comparison of network variants. They are however not appropriate for evaluating observation sets, which include several types of observations, e.g. spatial networks (plane observations and vertical ones). One of them is so called Otrebski's parameter, which is a measure of an average improvement in network accuracy as a result of adjusting the observations by the least squares method [8]. In the light of definition of the relevant theorem [3], this parameter (p) is expressed by means of the *a'priori* known sizes of data sets:

$$\frac{1}{m}\sum_{i=1}^{m}\frac{\hat{\mu}_i^2}{\mu_i^2}=\frac{n}{m}=p \tag{1}$$

where: μ_i^2, $\hat{\mu}_i^2$ – diagonal elements in observation covariance matrix before and after adjustment (respectively), m – number of observations, n – number of unknowns.

The p number is called Otrebski's parameter. In professional geodetic literature (e.g. in [7, 8]) this theorem is being proved in many various ways, for example by calculating the trace of the matrix product: $\mathbf{A} \cdot \mathbf{Q}_{\hat{L}}$, where: $\mathbf{P}$ – weight matrix of observations, $\mathbf{Q}_{\hat{L}}$ – covariance matrix of adjusted observations.

Thus to prove thesis of the theorem (1), it is enough to prove that

$$Tr\,\{\mathbf{P} \cdot \mathbf{Q}_{\hat{L}}\} = n \tag{2}$$

On the basis of definition [3] it can be said, that p parameter as a global parameter of network, characterizes the average network quality but only for the geometric elements observed.

Research work realised by author [2] has proved that: if an observation system is homogeneous (every point is determined by similar redundancy number), then the decrease in value of p parameter (more redundancy) involves an improvement in local accuracy characteristics. In general, if the number of unknowns n remains unchanged and the observation number m is increased by Δm, $(m_1 = m + \Delta m)$, then every accuracy characteristic of the network must be improved. The improvement of the local point parameters (e.g. mean errors of coordinates) depends on the connection between a point determined and the new observations. The change in the value of Otrebski's parameter follows then by the principle:

$$p - \Delta p = \frac{n}{m + \Delta m,} \tag{3}$$

hence

$$\Delta p = p \cdot \frac{\eta}{\eta + 1} \tag{4}$$

where: $\eta = \dfrac{\Delta m}{m}$ is a relative increase in number of observations.

Assuming that $\Delta m < < m$, the approximation can be used:

$$\frac{\Delta p}{p} = \frac{\eta}{\eta + 1} \approx \eta, \tag{5}$$

what means, that the relative decrease in value of p is comparable to the relative increase in number of observations m. Things look differently when comparing network variants, in which both number of observations and that of unknowns, are being changed (e.g. if adding new points to a net or converting the two-dimensional net to three-dimensional net). The condition of decrease in value of Otrebski's parameter results from the inequality:

$$\frac{n + \Delta n}{m + \Delta m} < \frac{n}{m} \tag{6}$$

It takes place only if

$$\frac{\Delta n}{\Delta m} < \frac{n}{m}, \tag{7}$$

that is, if the relative increase in number of unknowns is lower than the relative increase in number of observations. In all other situations Otrebski's parameter as a global criterion, does not decrease, that is, it either remains invariable

$$\frac{\Delta n}{\Delta m} = \frac{n}{m}, \tag{8}$$

or increases

$$\frac{\Delta n}{\Delta m} > \frac{n}{m}. \tag{9}$$

The last case (increasing in value of p) is a symptom of a disturbance in the network homogeneity.

The new observation system which is added, is internally worse than that one hitherto existing:

$$\frac{\Delta n}{\Delta m} = p_1 > p, \tag{10}$$

where: p_1 – internal Otrebski's parameter for the added observation system.

However, formula (10) does not mean that the local characteristics of network accuracy will be worsened. It can relate e.g. to the accuracy comparing of a horizontal network with the spatial one, which has been formed by including additional spatial observations (zenith angles, slope distances) in the horizontal network. If added observations bring any redundancy into network ($\Delta n < \Delta m$), then the decrease in the value of mean errors of horizontal coordinates must take place. However, behaviour of the global Otrebski's parameter will depend on the relation $\frac{\Delta n}{\Delta m}$.

From the above analysis it results that the use of Otrebski's parameter in its standard form (1) as a global criterion for comparison of accuracy variants of networks, is limited in principle to the case of homogenous network, even in a sense of dimension space.

But a question appears: is there a possibility of such modification of Otrebski's theorem, by means of which one could determine (with taking all observational data into consideration) the value of the relevant parameter (p) for the subset of unknowns characterized by a homogeneity of their determining ? It turns out there is such a possibility.

1. *Modification of Otrebski's theorem for equivalent observation system*

Theoretical base of the modification of Otrebski's parameter is the principle of elimination of the unknowns from observation equations and forming an equivalent system of conditional observations with unknowns. For better explanation I will use an example: There is a well known way of elimination of the orientation unknown from observation equations by summing up all the equations and subtracting the average equation from every observation equation. The new system of observation equations obtained is an equivalent system without orientation unknown. In general, the value of Otrebski's parameter can be calculated as follows:

$$p_1 = \frac{n}{m}, \tag{11}$$

$$p_2 = \frac{n - r}{m - r}, \tag{12}$$

where: p_1, p_2 – Otrebski's parameter before and after elimination of the orientation unknowns (respectively), r – number of station points equal to the number of orientation unknowns.

Parameter p_2 includes some information about evaluation of the network without orientation unknowns, so it only applies to the independent observations. As we are usually interested in determining the point coordinates, and not in determining the orientation unknowns, so the p_2 parameter seems to be more relevant criterion of evaluating the network quality.

The above example suggests the possibility of similar modification of Otrebski's theorem. In general we can consider elimination of any number r of parameters being determined. The modified Otrebski's parameter will be:

$$p(r) = \frac{n - r}{m - r}. \tag{13}$$

One can prove this formula in similar manner as in the classical case (2). Let us assume that in the initial system of m observation equations we can specify two groups of unknowns of sizes: $n - r$ and r (respectively). This system can be written as:

$$\mathbf{A}_1 \cdot \delta\mathbf{X}_1 + \mathbf{A}_2 \cdot \delta\mathbf{X}_2 = \delta\mathbf{L} + \mathbf{V}. \tag{14}$$

The initial matrix of coefficients $\mathbf{A}$ is divided into two sub-matrices:

$$\mathbf{A} = [\mathbf{A}_1 \vdots \mathbf{A}_2]. \tag{15}$$

Two vectors of unknowns $\delta\mathbf{X}_1$ and $\delta\mathbf{X}_2$ are characterized by sizes: $(n - r)$ and r (respectively). By eliminating r unknowns from this system we obtain a new system with $(m - r)$ equations of rank $r' = (n - r)$. It is the system of conditional equations with unknowns:

$$\mathbf{a} \cdot \delta\mathbf{X}_1 = \mathbf{l} + \mathbf{c} \cdot \mathbf{V} = \mathbf{l} + \mathbf{w}, \tag{16}$$

where: $\mathbf{a}$ – $(m\text{-}r) \times r'$ matrix of coefficients, $\mathbf{l}$ – transformed vector of free terms (pseudo-observations), $\mathbf{c}$ – $(m\text{-}r) \times m$ matrix, $\mathbf{l}$ – vector of corrections to pseudo-observations.

The covariance matrix for the equivalent system of pseudo-observations (in agreement with the law of propagation of variance) will be then expressed by formula:

$$\mathbf{Q}_{\mathrm{w}} = \mathbf{c} \cdot \mathbf{Q}_L \cdot \mathbf{c}^T. \tag{17}$$

The covariance matrix of vector of unknowns $\mathbf{X}_1$ can be written as:

$$\mathbf{Q}_{\mathbf{X}_1} = (\mathbf{a}^T \cdot \mathbf{Q}_{\mathrm{w}}^{-1} \cdot \mathbf{a})^{-1}. \tag{18}$$

The covariance matrix of adjusted vector of pseudo-observations ($\hat{\mathbf{l}} = \mathbf{l} + \mathbf{w}$) will be:

$$\mathbf{Q}_{\hat{\mathrm{l}}} = \mathbf{a} \cdot (\mathbf{a}^T \cdot \mathbf{Q}_{\mathrm{w}}^{-1} \cdot \mathbf{a})^{-1} \cdot \mathbf{a}^T. \tag{19}$$

For the determined covariance matrices, Otrebski's theorem can be proved in the same way as it was in the basic proof [3]:

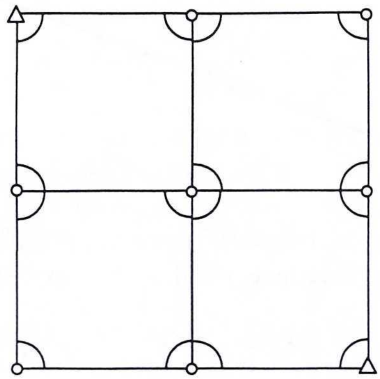
$$Tr\,\{\mathbf{Q}_{\mathrm{w}}^{-1} \cdot \mathbf{a} \cdot \mathbf{Q}_{\mathbf{X}_1} \cdot \mathbf{a}^T\} = Tr\,\{\mathbf{a}^T \cdot \mathbf{Q}_{\mathrm{w}}^{-1} \cdot \mathbf{a} \cdot \mathbf{Q}_{\mathbf{X}_1}\} = Tr\,\{\mathbf{I}\} = r', \tag{20}$$

where: $\mathbf{I}$ – unit matrix of size $[r' \times r']$.
Thus the modified Otrebski's theorem has been proved.

2. *An example of application of the new parameter*

There is an example of angular-linear network shown in Fig. 1. Let us assume that it includes 2 control points and 7 points to be determined. In the first case the net is treated as a two-dimensional construction, so the set of observations consists of 12 lengths and 15 independent horizontal angles. In the second case the horizontal net is supported by some additional vertical angles and now it is a three-dimensional net. Apart from the increase in number of observations, there is an increase in that of unknowns (additional altitudes).

The relation between the new observation number and that of unknowns in added observation set, complies with the condition (7).



o - station point
Δ - control point

Fig. 1. An example of the angular-linear network

In Table 1 there are compared both the characteristic quantities and value of *p* parameter for several variants of the observation set. It is worth noting that the new observations always cause an improvement in the quality of network (in a sense of *p* parameter), irrespective of whether the added set is internally better or worse than that hitherto existing.

T a b l e 1. Comparison of characteristic quantities for various variants of network

Ordinal No	Horizontal model of network					An observ. system added			Three dimensional network		$p_1 = \frac{n_l - r}{m_l - r}$
	Number of					Number of					
	lengths d	horizont. angles h_z	all observ. $m = d + h_z$	un-knowns n	$p = \frac{n}{m}$	vertical angles h_v (Δm)	altitudes r (Δn)	$p^l = \frac{\Delta n}{\Delta m}$	$m_l = m + h_v$	$n_l = n + r$	
1	2	3	4	5	6	7	8	9	10	11	12
1	12	15	27	14	0.52	-	-	-	-	-	-
2	12	15	27	14	0,52	15	9	0,60	42	25	0,42
3	12	15	27	14	0,52	19	9	0,47	46	25	0,38
4	12	15	27	14	0,52	24	9	0,38	51	25	0,33

CONCLUSIONS

The classical Otrebski's theorem, as a criterion of accuracy comparison of network variants, should be applied to the case of homogenous network. This restriction is related to the set of observations and set of unknowns as well as to dimension of space of analysed network.

However, this theorem can be modified by eliminating some elements from the observation equations. The elements being determined, that cause an inhomogeneity in the observation system, are being eliminated. It leads to forming up the equivalent system of conditional observations with unknowns. In this way, we obtain a modified Otrebski's theorem, which can be used e.g. to the research on reliability of a horizontal network supported with observations of vertical angles. In this case the altitude unknowns are being eliminated from the observation equations.

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Modyfikacja twierdzenia Otrębskiego dla ekwiwalentnych układów obserwacyjnych

S t r e s z c z e n i e

Praca zawiera propozycję modyfikacji twierdzenia Otrębskiego dla tak zwanych ekwiwalentnych układów obserwacyjnych. Modyfikacja prowadzi do utworzenia układu równań warunkowych z niewiadomymi. Przedstawiono również koncepcję zastosowania nowego parametru, jako globalnego kryterium oceny jakości sieci geodezyjnych cechujących się pewną niejednorodnością układu obserwacji.

Тадэуш Гаргуля

Модификация теоремы Отрембского для эквивалентных наблюдательных систем

Р е з ю м е

В работе представлено предложение модификации теоремы Отрембского для т.н. эквивалентных наблюдательных систем. Модификация приводит к сосданию системы условных уравнений с неизвестными, Представлена тоже концепция применения нового параметра как глобальной критерии оценки качества геодезических сетей, отличающихся некоторой неоднородностью системы наблюдений.