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INFLUENCE OF MATERIAL DEFECTS ON THE DYNAMIC STABILITY OF THE BERNOULLI-EULER BEAM

The paper presents the results of tests on dynamic stability of Bernoulli-Euler beam with damages. Damages (cracks) were modeled using three rotational springs. An analysis of the influence of crack depth and their position relative to the beam ends on dynamic stability of the beam was carried out. The problem of dynamic stability was solved by applying the mode summation method. Applying an orthogonal condition of eigenfunctions, the dynamic of the system was described with the use of the Mathieu equation. The obtained equation allowed the dynamic stability of the tested system to be analyzed. Stable and unstable solutions were analyzed using the Strutt card.

Keywords: material cracks, beam, dynamic stability, Mathieu equation

1. Introduction

The initiation and propagation of a crack in the material is most often the result of variable loads. Cracks reduce the stiffness of the elements and are the most common cause of structural defects. This type of structural component damage is a serious threat to the machine's proper functioning. Very often these elements are beams. There is a lot of work on testing beams with cracks. Chondros et al. in the work [1] presents the effect of transverse open crack on free vibrations of a simply supported Bernoulli-Euler beam. Kindova-Petrova in the work [2] presents the application of the vibration technique to detect the location and size of a fatigue crack in structures. The predicted crack depth and location are compared with data obtained from models obtained using FEM. The method of crack identification based on the assumption of transverse surface crack is presented by Rizos et al. in [3]. One-sided open crack was modeled with a rotational spring. Ostachowicz and Krawczuk in the study [4] presented an analysis of the impact of two open cracks on the frequencies of natural transverse vibrations in a cantilever beam. The cantilever beam with a crack was also analyzed by Barad, Sharma and Vyas in the paper [5]. Using the relationship between the rotational spring constant and the depth of the crack and its location, a method of detecting the crack was proposed. A crack in a structural element influence

on the vibration of the object and thus its dynamic stability. In the literature can find many works on the dynamic stability of the beams. Aldraihem and Baz considered the dynamic stability of beams with step changes in cross-section under moving loads [6]. The dynamic stability of the elastic beam was analyzed by Cederbaum and Mond in work [7]. In work [8] parametric instability of the electromagnetically excited beam was analyzed by Chen and Yeh. In the work [9]. Evensen and Evan-Iwanowski conducted analytical and experimental research on the impact of mass mounted at the end of a beam on its dynamic stability. In [10], dynamic stability of an elastically supported beam was analyzed (rotational and translational springs at the ends) by Majorana and Pellegrino. Sato, Saito and Otomi in [11] examined parametric vibrations of a horizontal beam loaded with concentrated mass, which showed the impact of beam weight and inertia of the rotational mass on beam vibrations. Sochacki in [12,13], a simply supported beam and a cantilever beam loaded axially with harmonic force were tested, demonstrating the destabilizing effect of additional elements attached to the beams.

This paper considers a simply supported beam loaded by a longitudinal force in the form $P(t) = P_0 + S \cos vt$. Additionally, the rotational springs modeling cracks there are on the beam at a chosen position between the supports. The considered beam was treated as Bernoulli – Euler beams. The problem of

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dynamic stability was solved using the mode summation method. The applied research procedure allowed the dynamics of the tested system to be described with the use of the Mathieu equation. The influence of the rotational springs (values of coefficients c) and their positions on the beam on the value of coefficient b in the Mathieu equation was investigated.

2. Mathematical model of beam vibrations

A diagram of the considered beam is presented in Fig. 1.

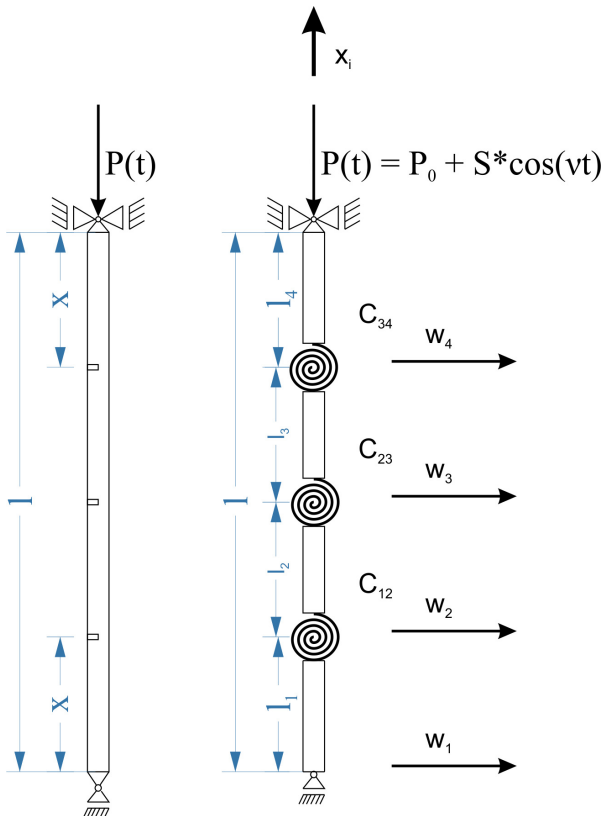


Fig. 1. Model of the beam with rotational springs modeling cracks

The vibration equation for i -th parts of a beam is known and has the following form:

$$E_i J_i \frac{\partial^4 w_{in}(x_i, t)}{\partial x_i^4} + P(t) \frac{\partial^2 w_{in}(x_i, t)}{\partial x_i^2} + \rho_i A_i \frac{\partial^2 w_{in}(x_i, t)}{\partial t^2} = 0 \quad (1)$$

where: $P(t) = P_0 + S \cos vt$, v – forcing frequency, ρ_i – density, A_i – cross-section area, $i = 1, 2, \dots, 4$ i -th part of the beam.

Substituting into equations (1)

$$w_{in}(x_i, t) = W_{in}(x_i) \cos(\omega_n t) \quad (2)$$

where: ω_n is n -th natural frequency of the beam, and into boundary conditions one can obtain (for $S = 0$):

$$E_i J_i W_{in}^{IV}(x_i) + P_0 W_{in}^{II}(x_i) - \rho_i A_i \omega_n^2 W_{in}(x_i) = 0 \quad (3)$$

and

$$W_1(0) = 0 \quad (4)$$

$$W_1^{II}(0) = 0 \quad (5)$$

$$W_i(l_i) = W_{i+1}(0) \quad (6)$$

$$C_{ii+1} (W_i^I(l_i) - W_{i+1}^I(0)) + E_i J_i W_i^{II}(l_i) = 0 \quad (7)$$

$$E_i J_i W_i^{II}(l_i) - E_{i+1} J_{i+1} W_{i+1}^{II}(0) = 0 \quad (8)$$

$$E_i J_i W_i^{III}(l_i) + P_0 W_i^I(l_i) - E_{i+1} J_{i+1} W_{i+1}^{III}(0) - P_0 W_{i+1}^I(0) = 0 \quad (9)$$

$$W_4(l_4) = 0 \quad (10)$$

$$W_4^{II}(l_4) = 0 \quad (11)$$

where the Roman numerals denote differentiation with respect to x and $i = 1, 2, 3$.

The general solution to equations (3) takes the form:

$$W_{in}(x_i) = D_{i1} \cosh(\alpha_{in} x_i) + D_{i2} \sinh(\alpha_{in} x_i) + D_{i3} \cos(\beta_{in} x_i) + D_{i4} \sin(\beta_{in} x_i) \quad (12)$$

where D_{ik} are integration constants ($k = 1, 2, \dots, 4$) and:

$$\alpha_{in}^2 = -\frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \Omega_{in}} \quad (13)$$

$$\beta_{in}^2 = \frac{\lambda_i}{2} + \sqrt{\frac{\lambda_i^2}{4} + \Omega_{in}} \quad (14)$$

where: $\Omega_{in}^2 = \omega_n^2 \frac{\rho_i A_i}{E_i J_i}$, $\lambda_i = \frac{P_0}{E_i J_i}$, $i = 1, 2, \dots, 4$

The equations of vibrations (3) together with the boundary conditions (4-11) allow the boundary value problem of the investigated beam to be formulated. The natural frequency ω_n and eigenfunctions of the beam $W_{in}(x_i)$ are determined by solving the boundary value problem.

3. The solution to the problem of the dynamic stability of the beam

The solution to equation (1) is assumed to be in the form of an eigenfunction series [9].

$$w_i(x_i, t) = \sum_{n=1}^{\infty} W_{in}(x_i) T_n(t) \quad (15)$$

where: $T_n(t)$ are unknown time functions and $W_{in}(x_i)$ are normalized eigenfunctions of free frequencies of i -th parts of the beam which satisfies

$$\sum_{i=1}^4 \rho_i A_i \int_0^{l_i} W_{in}(x_i) W_{im}(x_i) dx_i \begin{cases} = 0 & \text{gdy } m \neq n \\ = \gamma_m^2 & \text{gdy } m = n \end{cases} \quad (16)$$

Substituting solution (15) into equation (1) one can obtain:

$$\sum_{n=1}^{\infty} \left[\begin{aligned} &E_i J_i W_{in}^{IV}(x_i) T_{in}(t) + (P_0 + S \cos vt) \cdot \\ &\cdot W_{in}^{II}(x_i) T_{in}(t) + \rho_i A_i W_{in}(x_i) \ddot{T}_{in}(t) \end{aligned} \right] = 0 \quad (17)$$

After multiplying by $W_{im}(x_i)$, one can receive from equation (17):

$$\sum_{n=1}^{\infty} \left[\begin{aligned} &E_i J_i W_{in}^{IV}(x_i) W_{im}(x_i) T_n(t) + \\ &+ P_0 W_{in}^{II}(x_i) W_{im}(x_i) T_n(t) + \\ &+ S \cos vt W_{in}^{II}(x_i) W_{im}(x_i) T_n(t) + \\ &+ \rho_i A_i W_{in}(x_i) W_{im}(x_i) \ddot{T}_n(t) \end{aligned} \right] = 0 \quad (18)$$

From equations (3), after multiplying by $W_{im}(x_i)$, one can obtain:

$$\begin{aligned} E_i J_i W_{in}^{IV}(x_i) W_{im}(x_i) + P_0 W_{in}^{II}(x_i) W_{im}(x_i) = \\ = \rho_i A_i \omega_n^2 W_{in}(x_i) W_{im}(x_i) \end{aligned} \quad (19)$$

then (18) takes the following form:

$$\sum_{n=1}^{\infty} \left[\begin{aligned} &\rho_i A_i \omega_n^2 W_{in}(x_i) W_{im}(x_i) T_n(t) + \\ &+ S \cos vt W_{in}^{II}(x_i) W_{im}(x_i) T_n(t) + \\ &+ \rho_i A_i W_{in}(x_i) W_{im}(x_i) \ddot{T}_n(t) \end{aligned} \right] = 0 \quad (20)$$

Taking into account the basic parametric resonance, the first term of the sum from equation (20) was included in the further analysis. Thus, by integrating the expression (20) we obtained for the first term:

$$\begin{aligned} T_1(t) \left(\begin{aligned} &\omega_1^2 \rho_i A_i \int_0^l W_{i1}^2(x_i) dx_i + \\ &+ S \cos vt \int_0^l W_{i1}^{II}(x_i) W_{i1}(x_i) dx_i \end{aligned} \right) + \\ + \ddot{T}_1(t) \rho_i A_i \int_0^l W_{i1}^2(x_i) dx_i = 0 \end{aligned} \quad (21)$$

After appropriate transformations of the equation (21) and the substitution of t by a new variable $\tau = vt$, the following

form of the Mathieu equation was obtained (the subscripts were omitted).

$$\ddot{T}(\tau) + (a + b \cos \tau) T(\tau) = 0 \quad (22)$$

where: $a = \frac{\omega_1^2}{v^2}$, $b = \frac{S}{v^2} \frac{\sum_{i=1}^4 \int_0^l W_{i1}^{II}(x_i) W_{i1}(x_i) dx_i}{\sum_{i=1}^4 \rho_i A_i \int_0^l W_{i1}^2(x_i) dx_i}$, dots denote

differentiation with respect to τ .

The periodic solutions of the Mathieu equation (22) are known [14] and allow to define the limits of stable and unstable regions of the solutions (Strutt's chart).

4. Results and Discussion

In order to determine the dynamic stability of the analyzed system, parameter b of the Mathieu equation should be determined depending on the location of defects. In the study, the length of the analyzed beam was 3 m. The second crack was permanently in the middle of the beam, while the first and third cracks were symmetrical at a distance of x from the ends of the beam. Constant and variable part of longitudinal force was both 5% of critical load of the beam. Young's modulus was $2.1 \cdot 10^{11}$ Pa, the density of steel was equal to $7.86 \cdot 10^3$ kg · m⁻³. The cross-sectional area was $54 \cdot 10^{-3}$ m². The h is the percentage of crack depth in relation to the beam width.

Figure 2 shows the effect of the crack shift on the first and second natural frequencies. The value of the first natural frequency decreases as the first and third cracks shift towards the center of the beam. Increasing the crack depth results in greater differences in first natural frequency. On the other hand, the lowest values of the second natural frequency occur when the beam is divided equally into four parts. Similar frequency distribution for the first and second mode with increasing depth of the defect was obtained by Han-Ik Yoon et al. in [15].

Figure 3 shows the crack shift does not affect on the b -factor of Mathieu's equation. Kim and Kim in [16] showed that the location of the defect affects the dynamic stability, but they considered the situation with only one crack. This resulted in

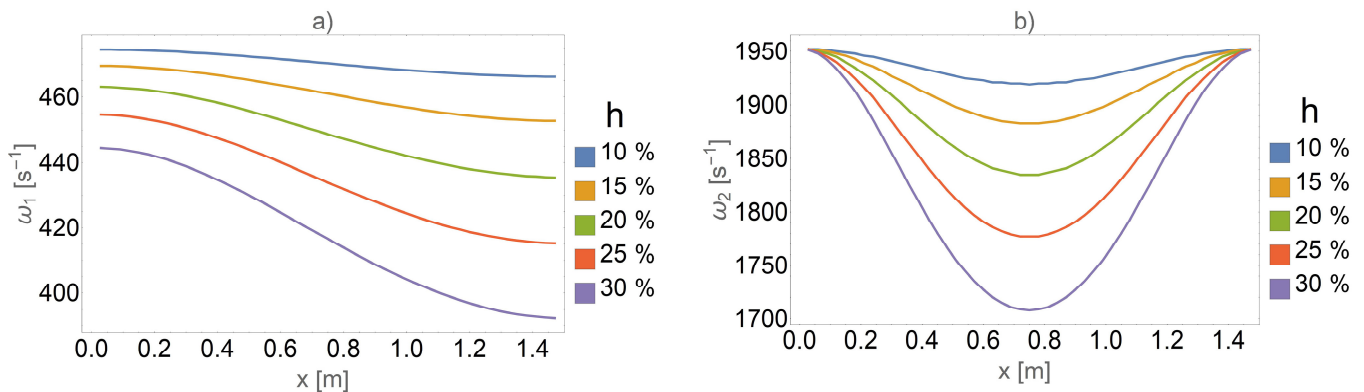


Fig. 2. Effect of crack shift x and size h on the first and second natural frequency

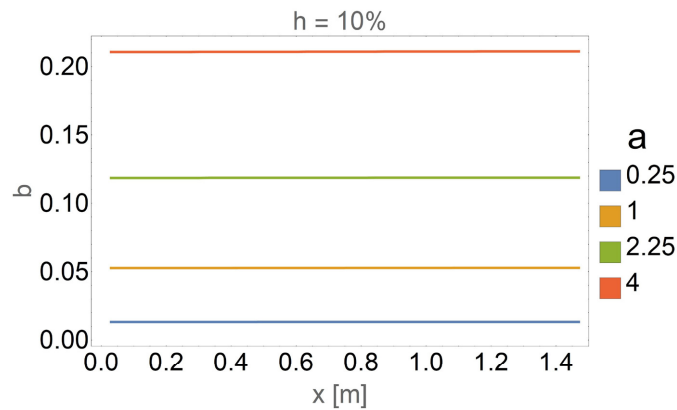


Fig. 3. Effect of crack shift on the b-factor of Mathieu's equation

an asymmetry of the beam. In the case analyzed, the beam and defect arrangement was always symmetrical, which resulted in the lack of changes in dynamic stability depending on the defect location. To ensure the legibility of the drawing, the values of the b coefficient for crack with a depth of 10% of the beam width were presented, but the values of the b coefficient were similar for the greater crack depths, which can be seen in Fig. 4. It should be noted that the defect depth above 40% significantly reduces the dynamic stability.

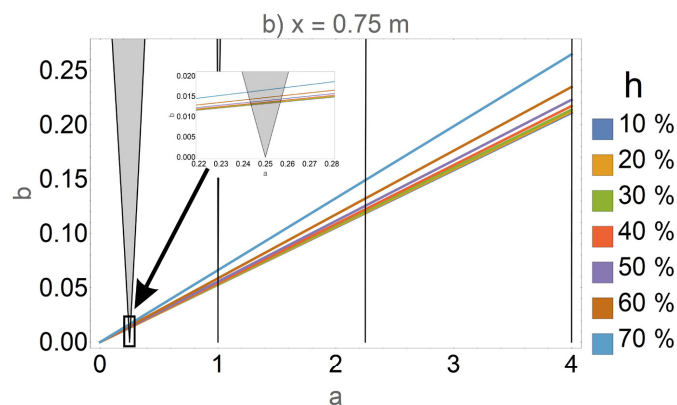


Fig. 4. Exemplary positions of solutions to the Mathieu equation for chosen values of crack depth

In Fig. 4, parameters a and b have been marked on the Ince-Strutt's card for the location of cracks from the ends of the beam $x = 0.25$ m. Unstable areas are marked in darker color. As in the work [16], an increase in instability was observed for deeper defects.

5. Conclusions

The paper shows that with increasing crack depth, the values of natural frequencies decrease. The first resonant frequency is at its lowest value when all the defects are closest to the center of the beam. The second, on the other hand, achieves a minimum when the cracks are evenly distributed over the beam. Due to the symmetry of the analyzed system, defects shifts do not change the dynamic stability. On the other hand, the crack depth above 40% of the beam thickness significantly reduces the beam's dynamic stability.

REFERENCES

- [1] T.G. Chondros, A.D. Dimarogonas, J. Yao, J. Sound Vib. **215** (1), 17-34 (1998), <https://doi.org/10.1006/jsvi.1998.1640>
- [2] D. Kindova-Petrova, J. Theor. Appl. Mech. Sofia **44** (4), 69-82 (2014), DOI: 10.2478/jtam-2014-0023
- [3] P.F. Rizos, N. Aspragathos, A.D. Dimarogonas, J. Sound Vib. **138** (3), 381-388 (1990), [https://doi.org/10.1016/0022-460X\(90\)90593-O](https://doi.org/10.1016/0022-460X(90)90593-O)
- [4] W.M. Ostachowicz, M. Krawczuk, J. Sound Vib. **150** (2), 191-201 (1991), [https://doi.org/10.1016/0022-460X\(91\)90615-Q](https://doi.org/10.1016/0022-460X(91)90615-Q)
- [5] K.H. Barad, D.S. Sharma, V. Vyas, Procedia Engineering **51**, 770-775 (2013).
- [6] O.J. Aldraihem, A. Baz, J. Sound Vib. **205** (5), 835-848 (2002), <https://doi.org/10.1006/jsvi.2001.3976>
- [7] G. Cederbaum, M. Mond, J. Appl. Mech. **59**, 16-19 (1992), <https://doi.org/10.1115/1.2899424>
- [8] C.-C. Chen, M.-K. Yeh, J. Sound Vib. **240** (4), 747-764 (2001), <https://doi.org/10.1006/jsvi.2000.3255>
- [9] H.A. Evensen, R.M. Evan-Iwanowski, J. Appl. Mech.-T. ASME **33**, 141-148 (1966), <https://doi.org/10.1115/1.3624971>
- [10] C.E. Majorana, C. Pellegrino, Engineering Computations **14** (7), 792-805 (1997), <https://doi.org/10.1108/02644409710188709>
- [11] K. Sato, V. Saito, V. Otomi, J. of Appl. Mech.-T. ASME **45** (3), 643-648 (1978), <https://doi.org/10.1115/1.3424375>
- [12] W. Sochacki, J. Sound Vib. **314** (1-2), 180-193 (2008), <https://doi.org/10.1016/j.jsv.2007.12.037>
- [13] W. Sochacki, J. Vibroengineering **15** (1), 280-290 (2013).
- [14] S.P. Timoshenko, V. Gere, Theory of Elastic Stability, Mc Graw-Hill - INC (1961).
- [15] Han-Ik Yoon, In-Soo Son, Sung-Jin Ahn, J. Mech. Sci. Technol. **21**, 476-485 (2007)
- [16] K.-H. Kim, J.-H. Kim, J. Sound Vib. **233** (1), 119-135. (2000). doi:10.1006/Jsvi.1999.2793