BULLETIN OF THE POLISH ACADEMY OF SCIENCES
TECHNICAL SCIENCES, Vol. 68, No. 3, 2020
DOI: 10.24425/bpasts.2020.133376

Positive extremal values and solutions of the exponential equations with application to automatics

H. GÓRECKI and M. ZACZYK*

AGH University of Science and Technology, Department of Automatics and Robotics, al. Mickiewicza 30, 30-059 Kraków, Poland

Abstract. In the paper, maximal values $x_e(\tau)$ of the solutions $x(t)$ of the linear differential equations excited by the Dirac delta function are determined. There are obtained the analytical solutions of the equations and also the maximal positive values of these solutions. The obtained sufficient conditions of the positivity of these solutions are defined by the Theorems. There are also formulated the necessary conditions of the positivity of these solutions. The analytical formulae enable the design of the system with prescribed properties [3].

Key words: extremal values, transfer function, transcendental equations.

1. Introduction

The extremal value of the state variable $x(t)$ has a fundamental role in many branches of the industry. In the chemical industry, the overrising temperature or pressure can lead to explosion. In the energy industry the overvoltage waves can destroy the installation. In the economic systems it is the determination of the maximal profit. The search for extremal values of the controlled quantity was the subject of many papers [5, 6, 10, 11, 12], however, no analytic formulae for their calculation were found. This paper provides original formulae which allow the determination of extremal values. For the first time, solutions of a certain class of transcendental equations are given in the form of analytical relationships. These formulae make it possible to estimate the accuracy of the performance of systems described by differential equations. In this way they fill the gap existing in the literature on the subject. In the article the theorems are proved, which give the analytical formulae for the determination of the maximal positive value of the state variable and the times in which these occur.

We consider the dynamic systems which are described by the differential equations

$$x^{(n)}(t)+a_1x^{(n-1)}(t)+\ldots+a_{n-1}x^{(1)}(t)+a_nx(t)= \\ =u^{(m)}(t)+b_1u^{(m-1)}(t)+\ldots+b_{m-1}u^{(1)}(t)+b_mu(t) \quad (1)$$

where:

a_i, b_i – constant parameters,

$x(t)$ – the dynamic error,

$u(t)=\delta(t)$ – Dirac impulse of the external signal.

The characteristic equation of the equation (1) is

$$M(s)=(s-s_1)(s-s_2)\ldots(s-s_n)=0. \quad (2)$$

We assume that the roots of the equation (2) are real, different and negative [4, p. 120]. We call them the poles.

$$0>s_1>s_2>\ldots>s_{n-1}>s_n\,. \quad (3)$$

The zeroes z_i of the polynomial $L(s)$ given below are real, different and negative.

$$L(s)=(s-z_1)(s-z_2)\ldots(s-z_m), \qquad m<n. \quad (4)$$

The essential assumption is that the poles of $M(s)$ and the zeroes of $L(s)$ interlace.

$$0>s_1>z_1>s_2>z_2\ldots>z_m>s_n\,. \quad (5)$$

We denote the transfer function

$$G(s)=\frac{L(s)}{M(s)} \quad (6)$$

and the solution of the equation (1) in the operational form is

$$\boldsymbol{X}(s)=\frac{\boldsymbol{L}(\boldsymbol{s})}{\boldsymbol{M}(\boldsymbol{s})}\delta(s). \quad (7)$$

Then the solution of the equation (1) in the time domain is

$$x(t)=\sum_{i=1}^{n}\frac{L(s_i)}{M^{(1)}(s_i)}e^{s_it}. \quad (8)$$

The initial conditions of the equation (1) depend on the Dirac impulse $\delta(t)$ and the poles and zeroes of the transfer function $G(s)$.

*e-mail: zaczyk@agh.edu.pl

Manuscript submitted 2019-12-22, revised 2020-01-27, initially accepted for publication 2020-02-20, published in June 2020

2. Statement of the problem

We want to determine the number of the extremal values $x(\tau)$ of the solutions $x(t)$, and the analytic formulae for the extremal times τ.

We consider the following four cases:

1. $L(s)=1$.
2. $L(s)=s-z_1$.
3. $L(s)=(s-z_1)(s-z_2)$.
4. $L(s)=(s-z_1)(s-z_2)\dots(s-z_{n-1})$.

3. Solution of the problem

3.1. Case 1. We assume $L(s)=1$. It means that $m=0$. The initial conditions of the equation (1) are forced by $\delta(t)$ [1]

$$\left.\begin{aligned} x^{(i)}(0)=0 \quad \text{for } i=0,1,\dots,n-2 \\ x^{(n-1)}(0)=1 \end{aligned}\right\}. \tag{9}$$

The solution of the equation (8) is

$$x(t)=\sum_{i=1}^{n}\frac{1}{M^{(1)}(s_i)}e^{s_it} \tag{10}$$

and the derivative

$$x^{(1)}(t)=\sum_{i=1}^{n}\frac{s_i}{M^{(1)}(s_i)}e^{s_it}. \tag{11}$$

The necessary condition for the extremum of $x(t)$ is

$$\sum_{i=1}^{n}\frac{s_i}{M^{(1)}(s_i)}e^{s_i\tau}=0. \tag{12}$$

In the work [2] it is proved that equation (12) has only one solution $\tau_{e_2}>0$.

Taking into account the initial conditions (9) we claim that the equation $x^{(1)}(t)=0$ has a solution of the multiplicity $(n-2)$ for $\tau_{e_1}=0$ and a single solution for $\tau_{e_2}>0$.

Theorem 1. If the roots of the characteristic equation (2) are located in the constant distance between them, that is

$$s_1,\;\; s_2=s_1+\Delta s,\;\; s_3=s_2+\Delta s,\;\; \dots,\;\; s_n=s_{n-1}+\Delta s \tag{13}$$

then the times τ of the extremums are equal

$$\left.\begin{aligned} \tau_{e_1}=\tau_{e_2}=\dots\tau_{e_{n-2}}=0 \\ \tau_{e_{n-1}}=\frac{n-1}{s_1-s_n}\ln\left(\frac{s_n}{s_1}\right) \end{aligned}\right\}. \tag{14}$$

Proof. We denote the coefficients of the equation (8)

$$\frac{L(s_i)}{M^{(1)}(s_i)}=A_i, \qquad i=1,2,\dots,n. \tag{15}$$

Multiplying equation (12) by e^{-s_nt} we have

$$\begin{aligned} s_1A_1e^{(s_1-s_n)t}+s_2A_2e^{(s_2-s_n)t}+\dots+ \\ +s_{n-1}A_{n-1}e^{(s_{n-1}-s_n)t}+s_nA_n=0. \end{aligned} \tag{16}$$

The coefficients

$$\left.\begin{aligned} B_1&=s_1A_1 \\ B_2&=s_2A_2 \\ &\vdots \\ B_n&=s_nA_n \end{aligned}\right\} \tag{17}$$

are equal to the corresponding coefficients of the equation

$$\left(e^t-1\right)^{n-2}\left(e^{\frac{(s_1-s_n)t}{n-1}}-\frac{s_n}{s_1}\right)=0 \tag{18}$$

which may be verified by inspection. □

Theorem 2. In the particular case when

$$s_1,\quad s_2=2s_1,\quad s_3=3s_1,\quad \dots,\quad s_n=ns_1 \tag{19}$$

it holds that

$$\left.\begin{aligned} \tau_{e_1}=\tau_{e_2}=\dots\tau_{e_{n-2}}=0 \\ \tau_{e_{n-1}}=-\frac{1}{s_1}\ln(n) \end{aligned}\right\}. \tag{20}$$

In the proof of Theorem 2 we have, according to the equation (18), the simple equation

$$\left(e^t-1\right)^{n-2}\left(e^{-s_1t}-n\right)=0. \tag{21}$$

For example, when: $s_1=-1$, $s_2=-2$, $s_3=-3$ we have

$$x^{(1)}(t)=-\frac{1}{2}e^{-t}+2e^{-2t}-\frac{3}{2}e^{-3t}=0. \tag{22}$$

From the necessary condition for extremum $x^{(1)}(t)=0$ after some manipulations we obtain from (22) the equation

$$x^{(1)}(t)=e^{2t}-4e^t+3=0. \tag{23}$$

From the equation (21) we have

$$\left(e^t-1\right)\left(e^t-3\right)=e^{2t}-4e^t+3=0 \tag{24}$$

which is identical with the equation (23).

Example

In Fig. 1 the time response of the system is shown for:

$$s_1=-1,\quad s_2=-3,\quad s_3=-5,\quad s_4=-7.$$

Numerical solution gives:

$\tau_e=(0,\ 0,\ 0.9729550745)$, $x_e=0.004958717576$.

From (14) we have: $\tau_e=0,\ 0,\ \frac{1}{2}\ln(7)=0.9729550745$.

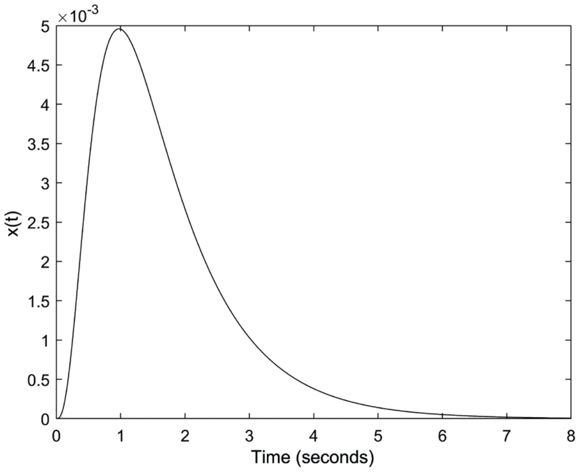

Fig. 1. Time response of the system for: $s_1 = -1$, $s_2 = -3$, $s_3 = -5$, $s_4 = -7$

In Fig. 2 the time response of the system is shown for:

$$s_1 = -1, \quad s_2 = -2, \quad s_3 = -3, \quad s_4 = -4.$$

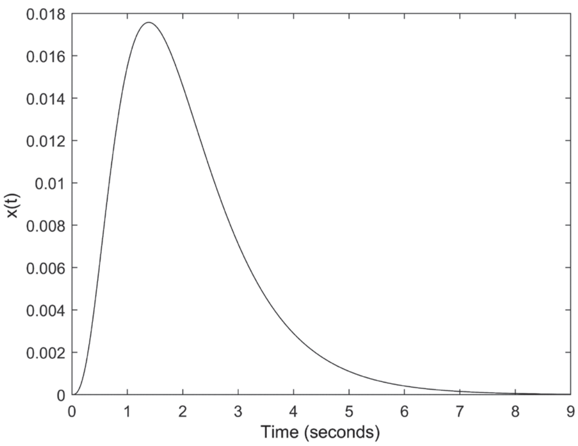

Fig. 2. Time response of the system for: $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $s_4 = -4$

Numerical solution gives:

$\tau_e = (0,\ 0,\ \ln(4))$, $x_e = 0.017578125$.

From (20) we have: $\tau_e = 0,\ 0,\ \ln(4)$.

3.2. Case 2.

We assume that

$$L(s) = s - z_1 \tag{25}$$

which means that $m = 1$, $z_1 < s_i < 0$ $(i = 1, 2, \ldots, n)$.

The initial conditions of the equation (1) in this case are

$$\left.\begin{aligned} &x^{(i)}(0) = 0 \quad \text{for } i = 0, 1, \ldots, (n-3) \\ &x^{(n-2)}(0) = 1 \\ &x^{(n-1)}(0) = \sum_{i=1}^{n} (s_i - z_1) = -a_1 - z_1 < 0 \end{aligned}\right\}. \tag{26}$$

The solution of equation (1) is

$$x(t) = \sum_{i=1}^{n} \frac{s_i - z_1}{M^{(1)}(s_i)} e^{s_i t} \tag{27}$$

then the derivative is

$$x^{(1)}(t) = \sum_{i=1}^{n} \frac{s_i(s_i - z_1)}{M^{(1)}(s_i)} e^{s_i t}. \tag{28}$$

The necessary condition of the extremum of $x(t)$ is

$$x^{(1)}(t) = 0. \tag{29}$$

Theorem 3. Taking into account the initial conditions (26) and the equation (29) we obtain the equation

$$\left(e^{s_1 t} - 1\right)^{n-3} \left[e^{2s_1 t} - \frac{(n+1)z_1 - (3n-1)s_1}{z_1 - s_1} e^{s_1 t} + {} + \frac{n(z_1 - ns_1)}{z_1 - s_1}\right] = 0. \tag{30}$$

The solutions of equation (30) are

$$\left.\begin{aligned} &t_{e_1}^{(n-3)} = 0 \\ &t_{e_2} = -\frac{1}{s_1} \ln\left[\frac{1}{2} \frac{(n+1)z_1 - (3n-1)s_1 \pm \sqrt{\Delta}}{z_1 - s_1}\right] \\ &\text{where} \\ &\Delta = [(n+1)z_1 - (3n-1)s_1]^2 - \\ &\quad -4[n(z_1 - ns_1)](z_1 - s_1) \end{aligned}\right\}. \tag{31}$$

Examples

In Fig. 3 the time response of the system is shown for:

$$s_1 = -1, \quad s_2 = -2, \quad s_3 = -3, \quad z_1 = -1.5.$$

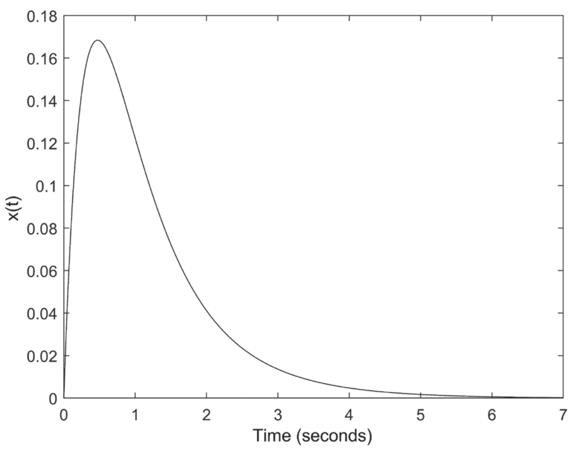

Fig. 3. Time response of the system for: $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $z_1 = -1.5$

Numerical solution gives:

$\tau_e = 0.4734671714$, $x_e = 0.168461248$.

From (31) we have:

$$\tau_e = \ln(1.605551276) = 0.4734671718.$$

In Fig. 4 the time response of the system is shown for:

$$s_1 = -1, \quad s_2 = -2, \quad s_3 = -3, \quad z_1 = -2.5.$$

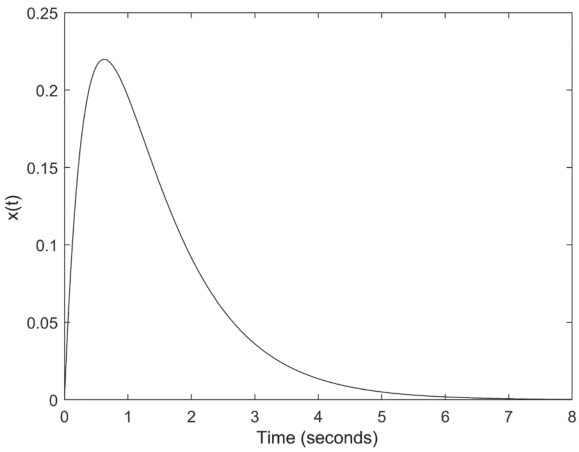

Fig. 4. Time response of the system for: $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $z_1 = -2.5$

Numerical solution gives:

$\tau_e = 0.6251451173$, $x_e = 0.2198549368$.

From (31) we have:

$$\tau_e = \ln(1.868517092) = 0.6251451173.$$

In Fig. 5 the time response of the system is shown for:

$$s_1 = -1, \quad s_2 = -2, \quad s_3 = -3, \quad z_1 = -3.5.$$

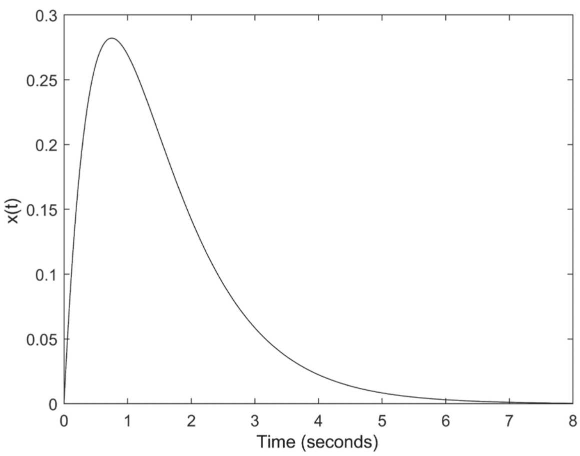

Fig. 5. Time response of the system for: $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $z_1 = -3.5$

Numerical solution gives:

$\tau_e = 0.7497709338$, $x_e = 0.2821127704$.

From (31) we have:

$$\tau_e = \ln(2.116515139) = 0.7497709338.$$

3.3. Case 3. We assume that

$$L(s) = (s - z_1)(s - z_2) \tag{32}$$

which means that $m = 2$,

$$s_i < z_2 < s_2 < z_1 < s_1 < 0, \qquad i = 3, 4, \ldots, n.$$

The initial conditions of the equation (1) in this case are

$$\left.\begin{array}{l} x^{(i)}(0) = 0 \quad \text{for } i = 0, 1, \ldots, (n-4) \\ x^{(n-3)}(0) = 1 \\ x^{(n-2)}(0) = \sum_{i=1}^{n} (s_i - (z_1 + z_2)) < 0 \\ x^{(n-1)}(0) = \sum_{i=1}^{n} (s_i - z_1)(s_i - z_2) > 0 \end{array}\right\}. \tag{33}$$

The solution of equation (1) is

$$x(t) = \sum_{i=1}^{n} \frac{(s_i - z_1)(s_i - z_2)}{M^{(1)}(s_i)} e^{s_i t}. \tag{34}$$

Then the derivative is

$$x^{(1)}(t) = \sum_{i=1}^{n} \frac{s_i(s_i - z_1)(s_i - z_2)}{M^{(1)}(s_i)} e^{s_i t}. \tag{35}$$

The necessary condition of the extremum $x(t)$ is

$$x^{(1)}(t) = 0. \tag{36}$$

Theorem 4. Taking into account the initial conditions (33) and the equations (35) and (36) we obtain the relation

$$\begin{aligned} &\left(e^{-s_1 t} - 1\right)^{n-4} \Big\{ [z_1 z_2 + z_1 + z_2 + 1] e^{-3s_1 t} - \\ &\quad - [(n+2)z_1 z_2 + 3nz_1 + 3nz_2 + 7n - 4] e^{-2s_1 t} + \\ &\quad + \left[(2n+1)z_1 z_2 + (n(n+3) - 1)z_1 + (n(n+3) - 1)z_2 + \right. \\ &\quad \left. + (2n+1)^2 + 2n(n-4)\right] e^{-s_1 t} - \\ &\quad - \left[nz_1 z_2 - n^2 z_1 - n^2 z_2 - n^3\right] \Big\} = 0. \end{aligned} \tag{37}$$

It can be shown that the time t_{e_2} is determined from the equation (37).

The proof is similar to that of Theorem 1. We state that in the case 1 the equation for the t_{e_2} was linear (21). In the case 2 with $z_1 < 0$ we obtain for t_{e_2} the equation of the 2-nd degree. It can be shown that t_{e_2} can be determined by the equation of the 3-rd degree when we have two zeroes z_1, z_2. Finally, when we have $z_1, z_2, \ldots, z_{n-1}$, it means that $m = n - 1$, and no time t_{e_2} can exist [see 2].

Examples

In Fig. 6 the time response of the system is shown for:

$$s_1 = -1,\ s_2 = -2,\ s_3 = -3,\ s_4 = -4,\ z_1 = -1.5,\ z_2 = -2.5.$$

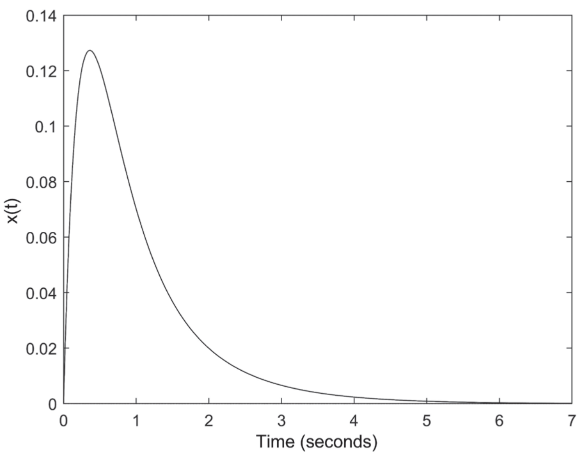

Fig. 6. Time response of the system for: $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $s_4 = -4$, $z_1 = -1.5$, $z_2 = -2.5$

Numerical solution gives: $\tau_e = 0.3615498986$.
The time t_e obtained from (37) for $n = 4$ is:

$t_e = 0.3615498986$.

In Fig. 7 the time response of the system is shown for:

$$s_1 = -1,\quad s_2 = -2,\quad s_3 = -3,\quad s_4 = -4,\quad s_5 = -5,$$
$$z_1 = -1.5,\quad z_2 = -2.5.$$

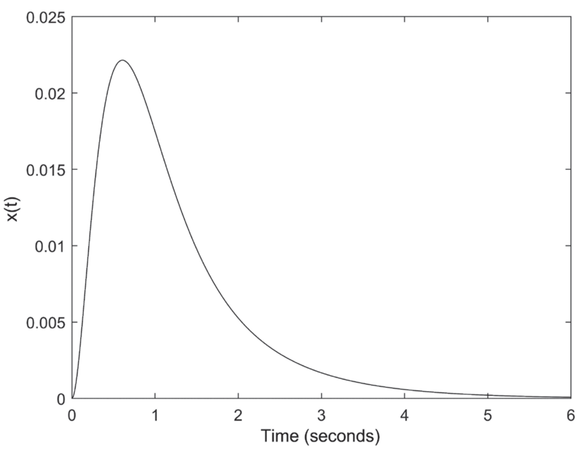

Fig. 7. Time response of the system for: $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $s_4 = -4$, $s_5 = -5$, $z_1 = -1.5$, $z_2 = -2.5$

Numerical solution gives: $\tau_e = 0.6076424554$.
The time t_e obtained from (37) for $n = 5$ is:

$t_e = 0.6076424551$.

In Fig. 8 the time response of the system is shown for:

$$s_1 = -1,\quad s_2 = -2,\quad s_3 = -3,\quad s_4 = -4,\quad s_5 = -5,\quad s_6 = -6,$$
$$z_1 = -1.5,\quad z_2 = -2.5.$$

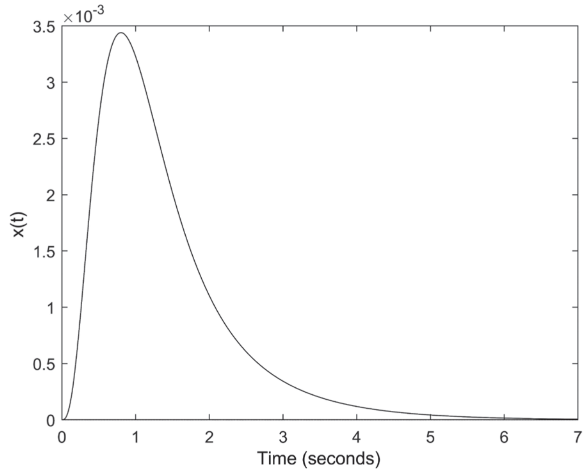

Fig. 8. Time response of the system for: $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $s_4 = -4$, $s_5 = -5$, $s_6 = -6$, $z_1 = -1.5$, $z_2 = -2.5$

Numerical solution gives: $\tau_e = 0.8016102738$.
The time t_e obtained from (37) for $n = 6$ is:

$t_e = 0.8016102730$.

3.4. Case 4.

Theorem 5. [2, 5]. In the case 4, we consider two polynomials

$$\begin{aligned} M(s) &= s^n + a_1 s^{n-1} + \ldots + a_{n-1}s + a_n = \\ &= (s - s_1)(s - s_2)\ldots(s - s_n), \\ L(s) &= s^m + b_1 s^{m-1} + \ldots + b_{m-1}s + b_m = \\ &= (s - z_1)(s - z_2)\ldots(s - z_m). \end{aligned}$$

We denote:

$$A_i = \frac{L(s_i)}{M^{(1)}(s_i)}, \qquad i = 1, 2, \ldots, n,$$

$$x(t) = \sum_{i=1}^{n} A_i e^{s_i t}.$$

If

1° $m = n - 1$,

$$2° \quad \left.\begin{aligned} &s_i < 0, \quad i = 1, 2, \ldots, n \\ &z_i < 0, \quad i = 1, 2, \ldots, n-1 \\ &s_i \neq s_j,\ z_i \neq z_j \quad \text{for } i \neq j \end{aligned}\right\},$$

3° the poles of $M(s)$ and the zeroes of $L(s)$ interlace

$$s_n < z_{n-1} < s_{n-1} < z_{n-2} < \ldots < s_2 < z_1 < s_1 < 0$$

then the function

$$x(t) = \sum_{i=1}^{n} A_i e^{s_i t}$$

has no extrema.

Proof. By the assumptions 1°, 2°, 3°, $A_i > 0$ $(i = 1, 2, \ldots, n)$.
From the assumption 3° it results that

$$\begin{cases} a_i > 0 \quad i = 1, 2, \ldots, n \\ b_i > 0 \quad i = 1, 2, \ldots, n-1 \\ M(0) > 0, \quad L(0) > 0 \\ M^{(1)}(s_1) > 0, \quad M^{(1)}(s_2) < 0, \quad M^{(1)}(s_3) > 0, \quad \ldots, \\ \qquad (-1)^{n-1} M^{(1)}(s_n) > 0 \\ L(s_1) > 0, \quad L(s_2) < 0, \quad L(s_3) > 0, \quad \ldots, \\ \qquad (-1)^{n-1} L(s_n) > 0. \end{cases}$$

From this we have that

$$A_i = \frac{L(s_i)}{M^{(1)}(s_i)} > 0, \qquad i = 1, 2, \ldots, n,$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sum_{i=1}^{n} s_i A_i e^{s_i t} < 0 \qquad \text{for each } t \in R.$$

The initial condition $x(0) = 1$ and $x(t)$ tends monotonically to zero as time $t \to \infty$ and no extremum exists. □

Example

In Fig. 9 the time response of the system is shown for:

$$s_1 = -1, \quad s_2 = -2, \quad s_3 = -3, \quad z_1 = -1.5, \quad z_2 = -2.5.$$

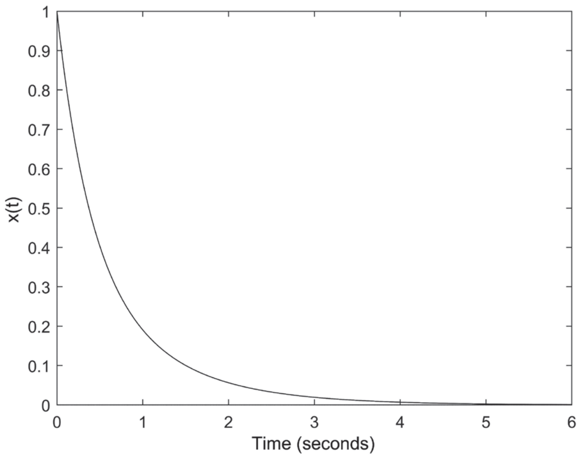

Fig. 9. Time response of the system for: $s_1 = -1$, $s_2 = -2$, $s_3 = -3$, $z_1 = -1.5$, $z_2 = -2.5$

The Theorem 1÷5 represent the sufficient conditions of the positive solutions of the state variable $x(t)$.

The necessary conditions of the positive solutions can be determined in the following way.

We consider the equation (1) with the right hand side equal zero:

$$a_0 x^{(n)}(t) + a_1 x^{(n-1)}(t) + \ldots + a_{n-1} x^{(1)}(t) + a_n x(t) = 0 \quad (38)$$

and the general initial conditions

$$x^{(i)}(0) = c_i \qquad \text{for } i = 0, 1, 2, \ldots, n-1. \quad (39)$$

We assume that if the solution of the equation (38) $x(t) \geq 0$ for $0 \leq t \leq \infty$, then the integral of the $x(t)$

$$J = \int_0^\infty x(t)\,\mathrm{d}t \quad (40)$$

is also positive.

Integrating each single term of the equation (38) we have

$$a_0 \int_0^\infty x^{(n)}(t)\,\mathrm{d}t + a_1 \int_0^\infty x^{(n-1)}(t)\,\mathrm{d}t + \ldots + \\ + a_{n-1} \int_0^\infty x^{(1)}(t)\,\mathrm{d}t + a_n \int_0^\infty x(t)\,\mathrm{d}t = 0. \quad (41)$$

From the equation (41) we obtain

$$a_0 \left[x^{(n-1)}(\infty) - x^{(n-1)}(0)\right] + a_1 \left[x^{(n-2)}(\infty) - x^{(n-2)}(0)\right] + \\ + \ldots + a_{n-1} \left[x^{(1)}(\infty) - x^{(1)}(0)\right] + \\ + a_n \int_0^\infty x(t)\,\mathrm{d}t = 0. \quad (42)$$

Taking into account the stable condition (3) that

$$x^{(i-1)}(\infty) = 0 \qquad \text{for } i = 1, \ldots, n \quad (43)$$

and the relation (42) and (40) we have that

$$J = \int_0^\infty x(t)\,\mathrm{d}t = \frac{a_0 c_n + a_1 c_{n-1} + \ldots + a_{n-1} c_1}{a_n}, \quad a_n > 0. \quad (44)$$

Theorem 6. If the solution of the equation (38) $x(t) \geq 0$ for $0 \leq t \leq \infty$ then the integral $J \geq 0$ for $0 \leq t \leq \infty$ and taking into account (44) we obtain the necessary condition for $x(t) \geq 0$

$$a_0 c_n + a_1 c_{n-1} + \ldots + a_{n-1} c_1 \geq 0, \qquad a_n \geq 0 \quad (45)$$

Unfortunately the necessary and sufficient conditions for the positive $x(t)$ are not yet known.

4. Conclusions

If the dynamic system is controlled by the Dirac impulse $\delta(t)$ and the poles and zeroes of the transfer function interlace, then only one extremum of $x(t)$ can exist for $m < n-1$. All the solutions are positive. For $m = n-1$ no extremum exists.

These results are very important in automation and in case of the long electrical lines. The positiveness of $x(t)$ is important in economics [6–10].

It is evident that with the increasing of the degree "n", the extremal value of time also increases $\tau = \ln(n)$ and the value of $x(\tau)$ diminishes.

References

[1] J. Osiowski, *An outline of operator calculus. Theory and applications in electrical engineering*, WNT, Warszawa, 1965 [in Polish].

[2] S. Białas, H. Górecki, and M. Zaczyk, "Extremal properties of the linear dynamic systems controlled by Dirac's impulse", *Int. J. Appl. Math. Comput. Sci*. 30 (1), 75–81 (2020).

[3] H. Górecki, and M. Zaczyk, "Design of systems with extremal dynamic properties", *Bull. Pol. Ac.: Tech*. 61 (3), 563–567 (2013).

[4] H. Górecki, *Optimization and Control of Dynamic Systems*, Springer, 2018.

[5] L. Farina, and S. Rinaldi, *Positive Linear Systems. Theory and Application*, J. Wiley, New York, 2000.

[6] T. Kaczorek, *Positive 1D and 2D Systems*, Springer-Verlag, London, 2002.

[7] T. Kaczorek, "A new method for determination of positive realizations of linear continuous-time systems", *Bull. Pol. Ac.: Tech*. 66 (5), 605–611 (2018).

[8] T. Kaczorek, "Global stability of nonlinear feedback systems with positive descriptor linear part", *Bull. Pol. Ac.: Tech.* 67 (1), 45–51 (2019).

[9] T. Kaczorek, "Stability of interval positive continuous-time linear systems", *Bull. Pol. Ac.: Tech.* 66 (1), 31–35 (2018).

[10] L. Benvenuti, and L. Farina, "A tutorial on the positive realization problem", *IEEE Trans. Autom. Control* 49 (5), 651–664 (2004)

[11] A. Rantzer, "On the Kalman-Yakubovich-Popov lemma for positive systems", *IEEE Trans. Autom. Control* 61 (5), 1346–1349 (2016).

[12] K. Sato and A. Takeda, "Construction methods of the nearest positive system", *IEEE Control Syst. Lett.* 4 (1), 97–102 (2020).