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Stability analysis of positive linear systems by decomposition of the state matrices into symmetrical and antisymmetrical parts

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Abstract. The stability of positive linear continuous-time and discrete-time systems is analyzed by the use of the decomposition of the state matrices into symmetrical and antisymmetrical parts. It is shown that: 1) The state Metzler matrix of positive continuous-time linear system is Hurwitz if and only if its symmetrical part is Hurwitz; 2) The state matrix of positive linear discrete-time system is Schur if and only if its symmetrical part is Hurwitz. These results are extended to inverse matrices of the state matrices of the positive linear systems.

Key words: linear, positive, system, decomposition, state matrix, stability.

1. Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in the monographs [1, 3, 7]. A variety of models having positive behavior can be found in engineering, especially in electrical circuits [15], economics, social sciences, biology and medicine, etc. [3, 7].

The positivity and stability of linear systems have been investigated in [2, 5–6, 8, 9, 16, 17, 22, 23] and of nonlinear systems in [10, 11]. A comparison of the stability of discrete-time and continuous-time linear systems has been given in [4]. The stability of interval positive linear systems with state matrices in integer and rational powers has been analyzed in [14]. The linear systems have been intensively investigated in [11, 12–15, 18–20] and the descriptor fractional linear systems with different fractional orders in [21].

In this paper the asymptotic stability of positive continuous-time and discrete-time linear systems by the decomposition of the state matrices into symmetrical and antisymmetrical parts will be addressed.

The paper is organized as follows. In Section 2 some preliminaries concerning positive continuous-time and discrete-time linear systems are recalled. Decomposition of the state matrices of positive linear continuous-time and discrete-time systems in symmetrical and antisymmetrical parts and the stability of the symmetrical parts are addressed in Section 3. The Comparison of the stability of positive continuous-time and discrete-time linear systems is presented in Section 4. Some concluding remarks are given in Section 5.

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The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ – the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n – the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n – the $n\times n$ identity matrix.

2. Preliminaries

Consider the continuous-time linear system

$$\dot{x} = Ax, \tag{1}$$

where $x = x(t) \in \Re^n$ is the state vector and $A = [a_{ij}] \in \Re^{n\times n}$.

Definition 1. [3, 7] The system (1) is called (internally) positive if $x(t) \in \Re_+^n$, $t \geq 0$ for all $x(0) \in \Re_+^n$.

Theorem 1. [3, 7] The system (1) is positive if and only if

$$A \in M_n. \tag{2}$$

Definition 2. [3, 7] The positive system (1) is called asymptotically stable (the matrix A is called Hurwitz) if

$$\lim_{t\to\infty} x(t) = 0 \ \text{ for all } \ x(0) \in \Re_+^n. \tag{3}$$

Theorem 2. [3, 7] The positive system (1) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1. All coefficients of the characteristic polynomial

$$\det[I_n s - A] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \tag{4}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

2. All principal minors $\overline{M}_i$, $i = 1, \ldots, n$ of the matrix $-A$ are positive, i.e.

$$\overline{M}_1 = |-a_{11}| > 0,\ \overline{M}_2 = \begin{vmatrix} -a_{11} & -a_{12} \\ -a_{21} & -a_{22} \end{vmatrix} > 0, \ldots, \\ \overline{M}_n = \det[-A] > 0. \tag{5}$$

3. There exists strictly positive vector $\lambda = [\lambda_1 \cdots \lambda_n]^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that

$$A\lambda < 0 \ \text{ and } \ \lambda^T A < 0. \tag{6}$$

Theorem 3. [15] If $A \in M_n$ is asymptotically stable then

$$-A^{-1} \in \Re_+^{n\times n}. \tag{7}$$

Consider the discrete-time linear system

$$x_{i+1} = Ax_1,\ \ i = 0, 1, \ldots \tag{8}$$

where x_i is the state vector and $A = [a_{ij}] \in \Re^{n\times n}$.

Definition 3. [3, 7] The system (8) is called (internally) positive if $x_i \in \Re_+^n$, $i \geq 0$ for all $x_0 \in \Re_+^n$.

Theorem 4. The system (8) is positive if and only if

$$A \in \Re_+^{n\times n}. \tag{9}$$

Definition 4. [3, 7] The positive system (8) is called asymptotically stable (the matrix A is called Schur) if

$$\lim_{t\to\infty} x_i = 0 \ \text{ for all } \ x_0 \in \Re_+^n. \tag{10}$$

Theorem 5. [3, 7] The positive system (8) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficients of the characteristic polynomial

$$\det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \ldots + \\ + a_1 z + a_0 \tag{11}$$

are positive, i.e. $a_i > 0$ for $i = 0, 1, \ldots, n-1$.

2) All principal minors $\hat{M}_i$, $i = 1, \ldots, n$ of the matrix $I_n - A$ are positive, i.e.

$$\hat{M}_1 = |1 - a_{11}| > 0,\ \hat{M}_2 = \begin{vmatrix} 1 - a_{11} & -a_{12} \\ -a_{21} & 1 - a_{22} \end{vmatrix} > 0, \\ \ldots, \hat{M}_n = \det[I_n - A] > 0. \tag{12}$$

3) There exists strictly positive vector $\lambda = [\lambda_1 \cdots \lambda_n]^T$, $\lambda_k > 0$, $k = 1, \ldots, n$ such that

$$A\lambda < \lambda \ \text{ and } \ \lambda^T A < \lambda^T. \tag{13}$$

Theorem 6. [15] If $A \in \Re_+^{n\times n}$ is Schur matrix then

$$(I_n - A)^{-1} \in \Re_+^{n\times n}. \tag{14}$$

Theorem 7. Let s_k, $k = 1, \ldots, n$ be the eigenvalues of the matrix $A \in \Re^{n\times n}$. Then:

1) $-s_k$, $k = 1, \ldots, n$ are the eigenvalues of the matrix $-A$,

2) s_k^{-1}, $k = 1, \ldots, n$ are the eigenvalues of the inverse matrix ($\det A \neq 0$) $A^{-1} \in \Re^{n\times n}$.

Proof. From the equality

$$-I_n[I_n\lambda - A] = [I_n(-\lambda) - (-A)] \tag{15}$$

it follows that $\det[I_n(-\lambda) - (-A)] = 0$ if and only if $\det[I_n\lambda - A] = 0$ since $\det[-I_n(I_n\lambda - A)] = (-1)^n \det[I_n\lambda - A]$.

If $\det A \neq 0$ then from the equality

$$-(A\lambda)^{-1}[I_n\lambda - A] = [I_n\lambda^{-1} - A^{-1}] \tag{16}$$

it follows that $\det[I_n\lambda^{-1} - A^{-1}] = 0$ if and only if $\det[I_n\lambda - A] = 0$ since

$$\det[-(A\lambda)^{-1}(I_n\lambda - A)] = \det[-(A\lambda)^{-1}]\det[I_n\lambda - A]$$

and $\det[A\lambda]^{-1} \neq 0$. □

Example 1. The matrix

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \tag{17}$$

with characteristic polynomial

$$\det[I_2\lambda - A] = \begin{vmatrix} \lambda + 2 & -1 \\ -1 & \lambda + 2 \end{vmatrix} = \lambda^2 + 4\lambda + 3 \tag{18}$$

has the eigenvalues $\lambda_1 = -1$, $\lambda_2 = -3$.

The matrix

$$-A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{19}$$

with characteristic polynomial

$$\det[I_2\lambda - (-A)] = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = \lambda^2 + 4\lambda + 3 \tag{20}$$

has the eigenvalues $\lambda_1 = 1$, $\lambda_2 = 3$.

The inverse matrix of (17) has the form

$$A^{-1} = \frac{1}{3}\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix} \tag{21}$$

and its characteristic polynomial

$$\det\left[I_2\lambda - A^{-1}\right] = \begin{vmatrix} \lambda + \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \lambda + \frac{2}{3} \end{vmatrix} = \lambda^2 + \frac{4}{3}\lambda + \frac{1}{3} \tag{22}$$

has the zeros $\lambda_1 = -1$, $\lambda_2 = -1/3$.

This confirms the Theorem 7.

3. Decomposition of the state matrices into symmetrical and antisymmetrical parts

Consider the Metzler matrix $A = [a_{ij}] \in M_n$ which is in general case not symmetrical. It is well-known the this matrix can be decomposed into symmetrical part A_s and the antisymmetrical part A_a

$$A = A_s + A_a \tag{23a}$$

where

$$A_s = \frac{A + A^T}{2}, \quad A_a = \frac{A - A^T}{2} \tag{23b}$$

and T denotes the transpose.

The following properties of the matrices (23b) are well-known:

1) All eigenvalues $\bar{s}_1, \ldots, \bar{s}_n$ of the matrix A_s are real and of the matrix $A_a = [\hat{a}_{ij}]$, $\hat{s}_1, \ldots, \hat{s}_n$ are imaginary.
2) The trace of the matrix A is

$$\mathrm{tr}A = \mathrm{tr}A_s = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \bar{s}_i \tag{24a}$$

and

$$\mathrm{tr}A_a = \sum_{i=1}^{n} \hat{a}_{ii} = \sum_{i=1}^{n} \hat{s}_i = 0. \tag{24b}$$

Example 2. The Metzler matrix

$$A = \begin{bmatrix} -4 & 4 \\ 2 & -5 \end{bmatrix} \in M_2 \tag{25a}$$

can be decomposed into

$$A_s = \frac{A + A^T}{2} = \begin{bmatrix} -4 & 3 \\ 3 & -5 \end{bmatrix}, \qquad A_a = \frac{A - A^T}{2} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{25b}$$

The eigenvalues of A_s are real $\bar{s}_1 = \frac{-9 + \sqrt{37}}{2}$, $\bar{s}_2 = \frac{-9 - \sqrt{37}}{2}$ and of A_a are imaginary $\hat{s}_1 = j$, $\hat{s}_2 = -j$ and $\mathrm{tr}A_s = \bar{s}_1 + \bar{s}_2 = -9$, $\mathrm{tr}A_a = \hat{s}_1 + \hat{s}_2 = 0$.

Note that the eigenvalues $s_1 = \frac{-9 + \sqrt{33}}{2}$, $s_2 = \frac{-9 - \sqrt{33}}{2}$ of the matrix A and of the matrix A_s, $\bar{s}_1$, $\bar{s}_2$ are different and satisfy the condition $\mathrm{tr}A_s = \mathrm{tr}A = -9 = \bar{s}_1 + \bar{s}_2$.

Theorem 8. The Metzler matrix $A \in M_n$ is Hurwitz if and only if its symmetrical part A_s is Hurwitz.

Proof. By Theorem 2 the matrix $A \in M_n$ is Hurwitz if and only if there exists a strictly positive vector $\lambda \in \Re_+^n$ such that

$$A\lambda < 0 \ \text{ and } \ \lambda^T A < 0. \tag{26}$$

Taking into account that $\left(\lambda^T A\right)^T = A^T\lambda < 0$ and (1, 2) we obtain

$$A_s\lambda = \frac{A + A^T}{2}\lambda = \frac{A\lambda + A^T\lambda}{2} < 0. \tag{27}$$

Therefore, the Metzler matrix A is Hurwitz if and only if the matrix A_s is Hurwitz. □

Remark 1. If the matrix $A = [a_{ij}] \in M_n$ satisfies the conditions

$$\sum_{j=1}^{n} a_{ij} < 0 \quad \text{for} \quad i = 1, \ldots, n \qquad \text{and} \quad \sum_{i=1}^{n} a_{ij} < 0 \quad \text{for} \quad j = 1, \ldots, n \tag{28}$$

then we may choose the strictly positive vector in the form $\lambda = [1, \ldots, 1]^T \in \Re_+^n$.

Example 3. The Metzler matrix

$$A = \begin{bmatrix} -4 & 2 & 1 \\ 3 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix} \tag{29}$$

is Hurwitz since its characteristic polynomial

$$\det[I_3 s - A] = \begin{vmatrix} s+4 & -2 & -1 \\ -1 & s+1 & 0 \\ -3 & -1 & s+5 \end{vmatrix} = s^3 + 12s^2 + 42s + 40 \tag{30}$$

has positive coefficients (Theorem 2).

Choosing for the matrix (29) $\lambda = [0.74 \ \ 0.73 \ \ 0.70]^T$ we obtain

$$A\lambda = \begin{bmatrix} -4 & 2 & 1 \\ 1 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix}\begin{bmatrix} 0.74 \\ 0.73 \\ 0.70 \end{bmatrix} = \begin{bmatrix} -0.80 \\ -1.45 \\ -0.55 \end{bmatrix} \tag{31a}$$

and

$$\lambda^T A = \begin{bmatrix} 0.74 & 0.73 & 0.70 \end{bmatrix} \begin{bmatrix} -4 & 2 & 1 \\ 1 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix} = \\ = \begin{bmatrix} -0.13 & -0.01 & -2.76 \end{bmatrix}. \tag{31b}$$

The symmetrical and antisymmetrical parts of the matrix (29) have the forms

$$A_s = \frac{A + A^T}{2} = \begin{bmatrix} -4 & 1.5 & 2 \\ 1.5 & -3 & 0.5 \\ 2 & 0.5 & -5 \end{bmatrix}, \\ A_a = \frac{A - A^T}{2} = \begin{bmatrix} 0 & 0.5 & -1 \\ -0.5 & 0 & -0.5 \\ 1 & 0.5 & 0 \end{bmatrix}. \tag{32}$$

The symmetrical part A_s is also Hurwitz since for $\lambda = [1\ 1\ 1]^T$

$$A_s \lambda = \begin{bmatrix} -4 & 1.5 & 2 \\ 1.5 & -3 & 0.5 \\ 2 & 0.5 & -5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -1 \\ -2.5 \end{bmatrix} \tag{33a}$$

and its characteristic polynomial

$$\det[I_3 s - A] = \begin{vmatrix} s+4 & -1.5 & -2 \\ -1.5 & s+3 & -0.5 \\ -2 & -0.5 & s+5 \end{vmatrix} = \\ = s^3 + 12s^2 + 40.5s + 32.75 \tag{33b}$$

has positive coefficients.

Note that the matrix (29) and its symmetrical part A_s have different characteristic polynomials.

To extend Theorem 8 to the discrete-time linear system we decomposed the state matrix of the system (8) into the symmetrical part

$$A_{ds} = \frac{A_d + A_d^T}{2} = \frac{A - I_n + A^T - I_n}{2} = A_s - I_n, \\ A_s = \frac{A + A^T}{2} \tag{34}$$

and the antisymmetrical part

$$A_{da} = \frac{A_d - A_d^T}{2} = \frac{A - I_n - A^T + I_n}{2} = \frac{A - A^T}{2}. \tag{35}$$

Theorem 9. The matrix A of the discrete-time system (8) is Schur matrix if and only if the matrix $A_s - I$ is Hurwitz.

Proof. Proof follows immediately from Theorems 8 and the decomposition of the matrix into the symmetrical and antisymmetrical parts. □

Example 4. The matrix

$$A = \begin{bmatrix} 0.4 & 0.2 \\ 0.4 & 0.6 \end{bmatrix} \tag{36}$$

has the symmetrical part

$$A_{ds} = \frac{A + A^T}{2} - I_2 = \begin{bmatrix} 0.4 & 0.3 \\ 0.3 & 0.6 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \\ = \begin{bmatrix} -0.6 & 0.3 \\ 0.3 & -0.5 \end{bmatrix} \tag{37a}$$

and the antisymmetrical part

$$A_{da} = \frac{A - A^T}{2} = \begin{bmatrix} 0 & -0.1 \\ 0.1 & 0 \end{bmatrix}. \tag{37b}$$

The matrix (37a) is Hurwitz Metzler matrix.

Example 5. The matrix

$$A = \begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \tag{38}$$

is Schur matrix since the characteristic polynomial

$$\det[I_3(z+1) - A] = \begin{vmatrix} z+0.6 & -0.1 & -0.2 \\ 0 & z+0.8 & -0.1 \\ -0.1 & -0.3 & z+0.5 \end{vmatrix} = \\ = z^3 + 1.9z^2 + 1.13z + 0.205 \tag{39}$$

of the matrix $A = [I_3]$ has the positive coefficients (condition 1 of Theorem 5). The same result we obtain for the matrix (38) using the condition 3 of Theorem 5 since for $\lambda = [1\ 1\ 1]^T$

$$\begin{bmatrix} 0.4 & 0.1 & 0.2 \\ 0 & 0.2 & 0.1 \\ 0.1 & 0.3 & 0.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.7 \\ 0.3 \\ 0.9 \end{bmatrix} < \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{40}$$

Decomposition of the matrix $A_d = A - I_n$ for (34) in the symmetrized and antisymmetrical parts yields

$$A_{ds} = \frac{A + A^T}{2} - I_n = \begin{bmatrix} 0.4 & 0.05 & 0.15 \\ 0.05 & 0.2 & 0.2 \\ 0.15 & 0.2 & 0.5 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.6 & 0.05 & 0.15 \\ 0.05 & -0.8 & 0.2 \\ 0.15 & 0.2 & -0.5 \end{bmatrix}. \tag{41a}$$

$$A_{da} = \frac{A - A^T}{2} - I_n = \begin{bmatrix} 0 & 0.05 & 0.05 \\ -0.05 & 0 & -0.1 \\ -0.05 & 0.1 & 0 \end{bmatrix}. \tag{41b}$$

The matrix (41a) is Hurwitz Metzler matrix and the matrix (38) is Schur matrix.

Now we shall extend the decomposition into the symmetrical and the antisymmetrical parts the inverse matrix A^{-1} to the Metzler Hurwitz matrix $A \in M_n$

$$A^{-1} = U + V, \tag{42a}$$

where

$$U = \frac{A^{-1} + (A^{-1})^T}{2}, \quad V = \frac{A^{-1} - (A^{-1})^T}{2}. \tag{42b}$$

Theorem 10. Let $A \in M_n$ be Hurwitz. Then

$$A_s^{-1}\lambda < 0 \text{ for any strictly positive } \lambda \in \Re_+^n. \tag{43}$$

Proof. By Theorem 3 $-A^{-1} \in \Re_+^{n\times n}$ and $A^{-2} = (-A^{-1})(-A^{-1}) \in \Re_+^{n\times n}$.

Premultiplying $A\lambda < 0$ by the matrix A^{-2} we obtain

$$A^{-2}A\lambda = A^{-1}\lambda < 0. \tag{44a}$$

Similarly, postmultiplying $\lambda^T A < 0$ by $A^{-2} \in \Re_+^{n\times n}$ we obtain

$$\lambda^T A A^{-2} < 0 \text{ and } \lambda^T A^{-1} < 0. \tag{44b}$$

Therefore, using (44) we obtain

$$U\lambda = \frac{A^{-1} + (A^{-1})^T}{2}\lambda < 0. \tag{45}$$

and the condition (43) is satisfied. □

Example 6. The Metzler matrix

$$A = \begin{bmatrix} -4 & 4 \\ 2 & -5 \end{bmatrix} \tag{46}$$

is Hurwitz since its characteristic polynomial

$$\det[I_2 s - A] = \begin{vmatrix} s+4 & -4 \\ -2 & s+5 \end{vmatrix} = s^2 + 9s + 12 \tag{47}$$

has positive coefficients.

The inverse matrix of (46)

$$A^{-1} = \begin{bmatrix} -4 & 4 \\ 2 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{5}{12} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} \end{bmatrix} \tag{48}$$

has all negative coefficients and $-A^{-1} \in \Re_+^{2\times 2}$.

Note that for $\lambda = [1 \;\; 0.8]^T$ we have

$$A\lambda = \begin{bmatrix} -4 & 4 \\ 2 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} -0.8 \\ -2 \end{bmatrix} < 0, \tag{49a}$$

$$A^{-1}\lambda = \begin{bmatrix} -\frac{5}{12} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} -\frac{41}{60} \\ -\frac{23}{30} \end{bmatrix} < 0 \tag{49b}$$

and

$$\lambda^T A^{-1} = [1 \;\; 0.8]\begin{bmatrix} -\frac{5}{12} & -\frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} -\frac{23}{60} & -\frac{9}{15} \end{bmatrix} < 0. \tag{49c}$$

The symmetrical and antisymmetrical parts of (48) have the forms

$$A_s^{-1} = \frac{A^{-1} + (A^{-1})^T}{2} = \begin{bmatrix} -\frac{5}{12} & -\frac{1}{2} \\ -\frac{1}{6} & -\frac{1}{3} \end{bmatrix}, \quad A_a^{-1} = \frac{A^{-1} - (A^{-1})^T}{2} = \begin{bmatrix} 0 & \frac{1}{6} \\ -\frac{1}{6} & 0 \end{bmatrix}. \tag{50}$$

and

$$A_s^{-1}\lambda = \begin{bmatrix} -\frac{5}{12} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{3} \end{bmatrix}\begin{bmatrix} 1 \\ 0.8 \end{bmatrix} = \begin{bmatrix} -\frac{41}{60} \\ -\frac{23}{30} \end{bmatrix} < 0. \tag{51}$$

Therefore, the relationship (51) confirms the Theorem 10.

4. Comparison of the stability of positive continuous-time and discrete-time linear systems

Theorem 11. If the eigenvalues λ_k, $k = 1, \ldots, n$ are the eigenvalues of the matrix $A \in \Re_+^{n \times n}$ satisfying the condition

$$\operatorname{Re}\lambda_k > 0 \quad \text{for} \quad k = 1, \ldots, n. \tag{52}$$

then the matrix

$$-A^{-1} \in M_n \tag{53}$$

is Hurwitz.

Proof. Proof will be accomplished by induction with respect to n. For $n = 1$ the hypothesis is evident since $A = a_{11} > 0$ and is $-A^{-1} = -1/a_1 \in M_1$ Hurwitz.

For $n = 2$ we have $A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \Re_+^{2 \times 2}$, $a_{ij} \geq 0$, $a_{11} > 0$, $a_{22} > 0$ since $\lambda_k > 0$, $k = 1, \ldots, n$ implies $\det A = a_{11}a_{22} - a_{12}a_{21} = \lambda_1\lambda_2 > 0$ and

$$\begin{aligned} -A_2^{-1} &= \frac{1}{\det A}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \\ &= \frac{1}{\det A}\begin{bmatrix} -a_{22} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix} \in M_2 \end{aligned} \tag{54}$$

is Hurwitz $\left(\operatorname{Re}(-\lambda_k)^{-1} < 0, k = 1, 2\right)$

Assume that $A_m \in \Re_+^{m \times m}$, $m = 2, 3, \ldots, n$ satisfies the condition (52). Using the extension method we shall show that $-A_m^{-1} \in M_m$ is Hurwitz. It is easy to verify that if

$$\begin{aligned} A_m &= \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1m} \\ a_{21} & a_{22} & \ldots & a_{2m} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mm} \end{bmatrix} = \begin{bmatrix} A_{m-1} & A_{12} \\ A_{21} & a_{mm} \end{bmatrix}, \\ A_{12} &= \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{m-1,m} \end{bmatrix}, \quad A_{21} = \begin{bmatrix} a_{m1} & a_{m2} & \ldots & a_{m,m-1} \end{bmatrix} \end{aligned} \tag{55}$$

then

$$\begin{aligned} A_m^{-1} &= \begin{bmatrix} A_{m-1}^{-1} + \dfrac{A_{m-1}^{-1}A_{12}A_{21}A_{m-1}^{-1}}{a_m} & -\dfrac{A_{m-1}^{-1}A_{12}}{a_m} \\ -\dfrac{A_{21}A_{m-1}^{-1}}{a_m} & \dfrac{1}{a_m} \end{bmatrix}, \\ a_m &= a_{mm} - A_{21}A_{m-1}^{-1}A_{12}. \end{aligned} \tag{56}$$

By assumption if $A_{m-1} \in \Re_+^{(m-1)\times(m-1)}$ and satisfies (52) then $-A_{m-1}^{-1} \in M_{m-1}$ is Hurwitz and from (56) we have $-A_m^{-1} \in M_m$ is Hurwitz since $A_{12} \in \Re_+^{m-1\times 1}$, $A_{21} \in \Re_+^{1\times m-1}$ and $a_m = a_{mm} - A_{21}A_{m-1}^{-1}A_{12} > 0$.

This completes the proof. □

Example 7. The matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 2 & 3 & 5 \\ 2 & 0 & 2 \end{bmatrix} \in \Re_+^{3\times 3} \tag{57}$$

has the characteristic polynomial

$$\begin{aligned} \det[I_3 s - A] &= \begin{vmatrix} s-2 & -1 & -3 \\ -2 & s-3 & -5 \\ 0 & 0 & s-2 \end{vmatrix} = \\ &= s^3 - 7s^2 + 14s - 8 \end{aligned} \tag{58}$$

and its zeros $s_1 = 1$, $s_2 = 2$, $s_3 = 4$ satisfy the condition (52).

The inverse matrix of (57) has the form

$$A^{-1} = \begin{bmatrix} \frac{3}{4} & -\frac{1}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \tag{59}$$

and its characteristic polynomial

$$\begin{aligned} \det[I_3 s - A^{-1}] &= \begin{vmatrix} s - \frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & s - \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & s - \frac{1}{2} \end{vmatrix} = \\ &= s^3 - \frac{7}{8}s^2 + \frac{7}{4}s - \frac{1}{8} \end{aligned} \tag{60}$$

has the zeros $s_1^{-1} = 1$, $s_2^{-1} = 1/2$, $s_3^{-1} = 1/4$.

From (59) we have the matrix

$$-A^{-1} = \begin{bmatrix} -\frac{3}{4} & \frac{1}{4} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \in M_3 \tag{61}$$

with the eigenvalues $\hat{s}_1 = -1$, $\hat{s}_2 = -1/2$, $\hat{s}_3 = -1/4$. Therefore, the matrix (61) is Hurwitz Metzler.

Theorem 12. Let the continuous-time linear system (1) with the matrix $A \in M_n$ be positive and Hurwitz. Then:
1) The system with the matrix $-A \notin M_n$ is not positive and not Hurwitz;
2) The system with the matrix $A^{-1} \notin M_n$ is not positive but Hurwitz;
3) The system with the matrix $-A^{-1} \in M_n$ is positive and not Hurwitz.

Proof.
1) If $A \in M_n$ then $-A \notin M_n$ (is not a Metzler matrix) and by condition 1 of Theorem 7 is not Hurwitz.
2) If $A \in M_n$ then $A^{-1} \notin M_n$ and by condition 2 of Theorem 7 is not Hurwitz.
3) If $A \in M_n$ then by Theorem 3 the matrix $-A^{-1} \in \Re_+^{n\times n}$ and it is a Metzler matrix. By condition 1) and 2) of Theorem 7 the eigenvalues of the matrix are located in the right-hand side of the complex plane and the system is unstable. □

Example 8. (continuation of Example 6).
The system with the matrix (46) is positive and Hurwitz and the matrix $-A$ has the form

$$-A = \begin{bmatrix} 4 & -4 \\ -2 & 5 \end{bmatrix} \notin M_n. \tag{62}$$

The corresponding system to (62) is not positive and unstable since its characteristic polynomial

$$\det[I_2 s - A] = \begin{vmatrix} s-4 & 4 \\ 2 & s-5 \end{vmatrix} = s^2 - 9s + 12 \tag{63}$$

has one negative coefficient.

The inverse matrix (46) has the form (48) and its coefficients are negative. Therefore, it is not a Metzler matrix.

The characteristic polynomial of the matrix (48)

$$\det[I_2 s - A^{-1}] = \begin{vmatrix} s+\frac{5}{12} & \frac{1}{3} \\ \frac{1}{6} & s+\frac{1}{3} \end{vmatrix} = s^2 + \frac{3}{4}s + \frac{1}{12} \tag{64}$$

has positive coefficients and the matrix is Hurwitz.

From (48) we have

$$-A^{-1} = \begin{bmatrix} \frac{5}{12} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix} \in M_2 \tag{65}$$

The matrix (65) is not Hurwitz since its characteristic polynomial

$$\det[I_2 s - A^{-1}] = \begin{vmatrix} s-\frac{5}{12} & -\frac{1}{3} \\ -\frac{1}{6} & s-\frac{1}{3} \end{vmatrix} = s^2 + -\frac{3}{4}s + \frac{1}{12} \tag{66}$$

has one negative coefficient.

Theorem 13. Let the discrete-time linear system (8) with the matrix $A \in \Re_+^{n\times n}$ be positive and Schur. Then:
1) The system with the matrix $-A \notin \Re_+^{n\times n}$ is not positive but the matrix is Schur;
2) The system with the matrix $A^{-1} \notin \Re_+^{n\times n}$ is not positive and the matrix is not Schur;
3) The system with the matrix $-A^{-1} \notin \Re_+^{n\times n}$ is not positive and the matrix is not Schur.

Proof.
1) If $A \in \Re_+^{n\times n}$ then $-A \notin \Re_+^{n\times n}$ and the system is not positive. By condition 1 of Theorem 7 if z_k is eigenvalue of A then $-z_k$ is the eigenvalue of $-A$. The discrete-time system is asymptotically stable since the stability of the discrete-time linear systems depends only on the moduli of the eigenvalues.
2) If $A \in \Re_+^{n\times n}$ then $A^{-1} \notin \Re_+^{n\times n}$ and the system is not positive. By condition 2 of Theorem 7 if z_k is the eigenvalue of A then z_k^{-1} is the eigenvalue of A^{-1}. The discrete-time system with the matrix A^{-1} is unstable since if $|z_k| < 1$ then $|z_k^{-1}| > 1$.
3) The system with the matrix $-A^{-1}$ is not positive since $-A^{-1} \notin \Re_+^{n\times n}$. The system is unstable since the eigenvalues of the matrix have moduli greater 1. □

Example 9. The discrete-time linear system (8) with the matrix

$$A = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} \in \Re_+^{2\times 2} \tag{67}$$

is positive and the matrix is Schur since the characteristic polynomial

$$\det[I_2(z+1) - A] = \begin{vmatrix} z+0.6 & -0.1 \\ -0.2 & z+0.7 \end{vmatrix} = \\ = z^2 + 1.3z + 0.4 \tag{68}$$

has positive coefficients.

The inverse matrix of (67) has the form

$$A^{-1} = \begin{bmatrix} 0.4 & 0.1 \\ 0.2 & 0.3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -1 \\ -2 & 4 \end{bmatrix} \notin \Re_+^{2\times 2} \tag{69}$$

and the characteristic polynomial

$$\det[I_2(z+1) - A^{-1}] = \begin{vmatrix} z+0.6 & 1 \\ 2 & z-3 \end{vmatrix} = \\ = z^2 - 5z + 4 \tag{70}$$

has one negative coefficient. Therefore, the system with the matrix (69) is not positive and unstable.

From (69) we have

$$-A^{-1} = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix} \notin \Re_+^{2\times 2} \tag{71}$$

and the characteristic polynomial

$$\det\left[I_2 z + A^{-1}\right] = \begin{vmatrix} z+3 & -1 \\ -2 & z+4 \end{vmatrix} = z^2 + 7z + 10 \tag{72}$$

has the zeros $z_1 = -2$, $z_1 = -5$. Therefore, the system with the matrix (71) is not positive and unstable.

The above considerations can be extended to linear systems with state matrices in integer and rational powers [13].

4. Concluding remarks

The stability of positive linear continuous-time and discrete-time systems has been analyzed by the use of the decomposition of the state matrices into symmetrical and antisymmetrical parts. It has been shown that the state Metzler matrix of positive continuous-time linear system is Hurwitz if and only if its symmetrical is Hurwitz (Theorem 8). Similarly the state matrix of positive linear discrete-time system is Schur if and only if its symmetrical part $A - I$ is Hurwitz Theorem 9). These results have been extended to inverse matrices of the state matrices of the positive linear systems (Theorem 10). A comparison of the stability of positive continuous-time and discrete-time linear systems has been given (Theorems 11, 12). The considerations have been illustrated by numerical examples of positive linear systems. The considerations can be extended to positive fractional linear systems.

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