

Heat conduction in a finite medium using the fractional single-phase-lag model

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Abstract. In the paper, a solution of the time-fractional single-phase-lagging heat conduction problem in finite regions is presented. The heat conduction equation with the Caputo time-derivative is complemented by the Robin boundary conditions. The Laplace transform with respect to the time variable and an expansion in the eigenfunctions series with respect to the space variable was applied. A method for the numerical inversion of the Laplace transforms was used. Formulation and solution of the problem cover the heat conduction in a finite slab, hollow cylinder and hollow sphere. The effect of the fractional order of the Caputo derivative and the phase-lag parameter on the temperature distribution in a slab has been numerically investigated.

Key words: single-phase-lagging heat conduction, Caputo derivative, fractional heat conduction.

1. Introduction

The classical theory of the heat conduction is based on the Fourier law [1]

$$\mathbf{q}(\mathbf{r}, t) = -k\nabla T(\mathbf{r}, t) \quad (1)$$

where \mathbf{q} is the heat flux vector, \mathbf{r} is the point in the considered region, t is the time, k is the thermal conductivity of the material, ∇ is the gradient operator and T is the temperature. The Fourier law assumes simultaneous appearance of the heat flux and the temperature gradient. This assumption implies an unrealistic infinitely fast heat flow in the medium. To avoid the non-physical speed of the heat signal in the mathematical model of the heat conduction, Tzou in paper [2] has been proposed a generalization of the Fourier law by introducing a phase-lag parameter. This leads to the heat conduction model which is called the single-phase-lag model of the heat conduction. In this case, the constitutive equation (1) is replaced by the following relationship

$$\mathbf{q}(\mathbf{r}, t + \tau) = -k\nabla T(\mathbf{r}, t) \quad (2)$$

where τ is the phase-lag parameter. Expanding the left-hand side of equation (2) into the fractional Taylor series with respect to variable τ and taking into account two terms of this series, we obtain the constitutive equation in the form [3]

$$\mathbf{q}(\mathbf{r}, t) + \frac{\tau^\alpha}{\Gamma(1+\alpha)} {}_0D_t^\alpha \mathbf{q} = -k\nabla T(\mathbf{r}, t), \quad 0 < \alpha \leq 1 \quad (3)$$

where Γ is the Gamma function and ${}_0D_t^\alpha$ denotes the left Caputo derivative of order α with respect to variable t . The left Caputo derivative ${}_aD_t^\alpha$ is defined by

$${}_aD_t^\alpha f(x, t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{1}{(t-u)^{\alpha+1-m}} \frac{\partial^m f(x, u)}{\partial u^m} du, & \text{for } m-1 < \alpha < m \\ \frac{\partial^m f(x, t)}{\partial t^m}, & \text{for } \alpha < m \in \mathbb{N} \end{cases} \quad (4)$$

The heat conduction equation can be derived by using the constitutive equation (3) and the energy conservation equation [1]

$$-\nabla \cdot \mathbf{q}(\mathbf{r}, t) + g(\mathbf{r}, t) = \rho C_p \frac{\partial T}{\partial t} \quad (5)$$

where $g(\mathbf{r}, t)$ is the volumetric rate of the heat generation, ρ is the density of the material and C_p is the specific heat capacity. Combining the constitutive equation (3) with the energy conservation equation (5) leads to the fractional single-phase-lagging heat conduction equation. In this approach the following property of the Caputo derivative is used [4]

$${}_aD_t^\alpha ({}_aD_t^m f(t)) = {}_aD_t^{\alpha+m} f(t) \quad \text{for } m = 0, 1, 2, \dots \quad (6)$$

The analytical solution of the fractional differential equations occurring in mathematical models can be obtained only for some particular cases of the considered problems. The numerical methods for solving the fractional problems have been applied by many authors (for example papers [5–6]). An application of the Laplace transform approach allows one to obtain a solution in the Laplace transform domain. For some

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problems, the analytical form of the inverse transforms can be obtained (see [7–8]). If an analytical form of the inverse Laplace transforms cannot be determined, a method for the numerical inversion may be applied [9–10]. A review of numerical methods to the inverse Laplace transform is given in paper [11]. An application of numerical inversion of Laplace transforms in fractional calculus was presented in paper [12].

Many researchers, for investigations of the heat transfer in a medium, use the dual-phase-lag models [13–14]. A particular case of these types models of heat transfer is the single-phase-lag model. The heat conduction model based on the fractional single-phase-lagging approach has been used in paper [15]. The solution of the problem was obtained by the Laplace transform technique. The effect of the fractional order on temperature in the thin film was investigated.

In this paper, the problem of the heat conduction in a finite region, based on the single-phase-lag model with the fractional time-derivative in the heat conduction equation, has been studied. The solution of the problem in the Laplace transform domain concerns the heat conduction in a slab, hollow cylinder and hollow sphere. The inverse of the Laplace transforms using the numerical method was determined. Numerical results are compared with an analytical solution which was appointed for a fixed value of the derivative order. The presented results of computations concern heat conduction in a slab with a heat source in the form of a ramp-type function and the heat conduction when the ambient temperature changes sinusoidally. The effect of the fractional derivative order and the phase-lag parameter on the temperature distribution in the slab has been investigated.

2. Formulation of the problem

The heat conduction equation can be derived using constitutive equation (3) and energy equation (5). The obtained fractional single-phase-lagging equation can be written in the form

$$\begin{aligned} \tau_{\alpha 0} D_t^{\alpha+1} T(\mathbf{r}, t) + \frac{\partial T}{\partial t} = \\ = \kappa \nabla^2 T(\mathbf{r}, t) + \frac{\kappa}{k} (g(\mathbf{r}, t) + \tau_{\alpha 0} D_t^{\alpha} g(\mathbf{r}, t)) \end{aligned} \quad (7)$$

where $\tau_{\alpha} = \tau^{\alpha}/\Gamma(1 + \alpha)$ and $\kappa = k/\rho C_p$ is the thermal diffusivity. This time-fractional equation is a type of the generalized Cattaneo equation which describes an anomalous transport processes [16]. According to definition (4), the time-fractional derivative depends on the past state, therefore equation (7) presents the description of the non-local heat conduction [17].

Equation (7) is valid in a region which is specified by the considered medium. We consider the heat conduction problem in a slab, cylinder and sphere. In the each of three cases, we assume the one dimensional heat conduction (for convenience, we assume $\mathbf{r} = x$): for the slab – in the direction of x -axis of the rectangular coordinate system, for the cylinder – in the radial

direction of the cylindrical coordinate system assuming axial symmetry and for the sphere – in the radial direction of the spherical coordinate system assuming the central symmetry. Therefore, the operator ∇^2 in equation (7) has the form [1]

$$\nabla^2 T = \frac{1}{x^p} \frac{\partial}{\partial x} \left(x^p \frac{\partial T}{\partial x} \right), \quad p = 0, 1, 2 \quad (8)$$

where $p = 0$ for the slab, $p = 1$ for the cylinder and $p = 2$ for the sphere.

Equation (7) is complemented by boundary and initial conditions. We assume the Robin boundary conditions for the slab, hollow cylinder and hollow sphere:

$$k \frac{\partial T}{\partial x}(a, t) = -h_a (T_a(t) - T(a, t)) \quad (9)$$

$$k \frac{\partial T}{\partial x}(b, t) = -h_b T(b, t) \quad (10)$$

and the initial conditions in the form

$$T(x, 0) = f(x), \quad \frac{\partial T}{\partial t}(x, 0) = h(x) \quad (11)$$

where h_a and h_b are convective heat transfer coefficients and $T_a(t)$ is the ambient temperature.

3. Solution of the problem

We search a solution of the problem (7, 9–11) in the form of a sum

$$T(x, t) = T_a(t) w(x) + \theta(x, t) \quad (12)$$

where

$$\theta(x, t) = \sum_{i=1}^{\infty} \Lambda_i(t) \Phi_i(x) \quad (13)$$

and $w(x)$ is defined as

$$w(x) = \begin{cases} \frac{1}{M_0} \left(\frac{k}{h_b} + b - x \right) & \text{for } p = 0 \\ \frac{1}{M_1} \left(\frac{k}{bh_b} - \ln \frac{x}{b} \right) & \text{for } p = 1 \\ \frac{1}{M_2} \left(\frac{k}{b^2 h_b} - \frac{1}{b} + \frac{1}{x} \right) & \text{for } p = 2 \end{cases} \quad (14)$$

whereas

$$M_0 = b - a + \left(\frac{1}{h_a} + \frac{1}{h_b} \right) k, \quad M_1 = \ln \frac{b}{a} + \left(\frac{1}{ah_a} + \frac{1}{bh_b} \right) k,$$

$$M_2 = \frac{1}{a} - \frac{1}{b} + \left(\frac{1}{a^2 h_a} + \frac{1}{b^2 h_b} \right) k.$$

The functions $\Phi_i(x)$ are solutions of the following eigenproblem

$$\frac{1}{x^p} \frac{\partial}{\partial x} \left(x^p \frac{\partial \Phi_i(x)}{\partial x} \right) + \lambda_i^2 \Phi_i(x) = 0 \tag{15}$$

$$k \frac{\partial \Phi_i}{\partial x}(a) - h_a \Phi_i(a) = 0 \tag{16}$$

$$k \frac{\partial \Phi_i}{\partial x}(b) + h_b \Phi_i(b) = 0 \tag{17}$$

Using the general solution of differential equation (13) and utilizing conditions (16–17), an equation serving to designate eigenvalues $\lambda_i, i = 1, 2, \dots$, is obtained. The eigenfunctions $\Phi_i(x)$ corresponding to determined eigenvalue λ_i , satisfy the orthogonality condition in the form

$$\int_a^b x^p \Phi_i(x) \Phi_j(x) dx = \begin{cases} 0 & \text{for } i \neq j \\ N_i & \text{for } i = j \end{cases} \tag{18}$$

where N_i is the square norm of the i -th eigenfunction defined by

$$N_i = \int_a^b x^p (\Phi_i(x))^2 dx. \tag{19}$$

Applying the orthogonality condition (18) to equation (13), we find the functions $\Lambda_i(t)$ as

$$\Lambda_i(t) = \frac{1}{N_i} \int_a^b x^p \Phi_i(x) \theta(x, t) dx. \tag{20}$$

In the further consideration we will use the relationship obtained by using equations (9–10) and (15–17). We write this relationship in the form

$$\int_a^b x^p \Phi_i(x) \nabla^2 \theta(x, t) dx = -\lambda_i^2 \int_a^b x^p \Phi_i(x) \theta(x, t) dx. \tag{21}$$

To derive an equation which will be used to determine the functions $\Lambda_i(t)$, we multiply equation (7) by the function $x^p \Phi_i(x)$ and integrate over the interval $[a, b]$ then we utilize the relationship (21). As a result, we find the time-fractional differential equation in the form

$$\tau_{\alpha 0} D_t^{\alpha+1} \Lambda_i + \frac{d\Lambda_i}{dt} + \kappa \lambda_i^2 \Lambda_i = F_i(t) \tag{22}$$

where

$$F_i(t) = \frac{\kappa}{N_i k} \int_a^b x^p \Phi_i(x) (\tau_{\alpha 0} D_t^{\alpha} g(x, t) + g(x, t)) dx - \frac{1}{N_i} \left(\tau_{\alpha 0} D_t^{\alpha+1} T_a(t) + \frac{dT_a(t)}{dt} \right) \int_a^b x^p \Phi_i(x) w(x) dx. \tag{23}$$

Similarly, multiplying both sides of equations (11) by $x^p \Phi_i(x)$ and integrating over the interval $[a, b]$, the following initial conditions are obtained

$$\Lambda_i(0) = \frac{1}{N_i} \int_a^b x^p \Phi_i(x) [f(x) - T_a(0) w(x)] dx \tag{24}$$

$$\frac{d\Lambda_i}{dt}(0) = \frac{1}{N_i} \int_a^b x^p \Phi_i(x) \left[h(x) - \frac{dT_a(0)}{dt} w(x) \right] dx.$$

3.1. Solution of the time-fractional equation in the Laplace transform domain.

We determine a solution of the initial problem (22–24) by using the Laplace transform technique. The Laplace transform $L[f(t)] = \bar{f}(s)$ of a function $f(t)$ is defined as

$$\bar{f}(s) = \int_0^{\infty} f(t) e^{-st} dt \tag{25}$$

where s is a complex parameter. We utilize the property of linearity of the Laplace transform and the following rule [18]

$$L\{ {}_0 D_t^{\alpha} f(x, t) \} = s^{\alpha} \bar{f}(x, s) - \sum_{k=0}^{m-1} s^{\alpha-1-k} f^{(k)}(x, 0^+), \tag{26}$$

$m - 1 < \alpha \leq m.$

After applying the Laplace transformation to equation (22) and using initial conditions (24), the Laplace transform of the solution of the problem (22–24) can be written in the form

$$\begin{aligned} \bar{\Lambda}_i = & \frac{(\tau_{\alpha} s^{\alpha} + 1)}{N_i (\tau_{\alpha} s^{\alpha+1} + s + \kappa \lambda_i^2)} \int_a^b x^p \Phi_i(x) \left(\frac{\kappa}{k} \bar{g}(x, s) + \right. \\ & \left. + f(x) \right) dx - \frac{\tau_{\alpha} s^{\alpha-1}}{N_i (\tau_{\alpha} s^{\alpha+1} + s + \kappa \lambda_i^2)} \int_a^b x^p \Phi_i(x) \cdot \\ & \cdot \left(\frac{\kappa}{k} g(x, 0) - h(x) \right) dx - \frac{s(\tau_{\alpha} s^{\alpha} + 1)}{N_i (\tau_{\alpha} s^{\alpha+1} + s + \kappa \lambda_i^2)} \cdot \\ & \cdot \bar{T}_a(s) \int_a^b x^p \Phi_i(x) w(x) dx. \end{aligned} \tag{27}$$

We further assume that the function $T_a(t)$ occurring in equation (27) has the form

$$T_a(t) = \hat{T}_a(1 + \sin \omega t). \tag{28}$$

Moreover, the volumetric rate of the heat generation $g(x, t)$ is given in the form of a ramp-type function

$$g(x, t) = (t - (t - t_1) H(t - t_1)) \frac{G_1}{t_1} \tag{29}$$

where $H(t - t_1)$ is the Heaviside function. Taking into account the Laplace transforms of functions (28) and assuming $h(x) = 0$

in equation (27), we present the inverse Laplace transform $\Lambda_i(t)$ as

$$\begin{aligned} \Lambda_i(t) = & -\frac{\hat{T}_a}{N_i} (U_i(t) + V_i(t)) \int_a^b x^p \Phi_i(x) w(x) dx + \\ & + \frac{1}{N_i} V_i(t) \int_a^b x^p \Phi_i(x) f(x) dx + \frac{\kappa G_1}{t_1 k N_i} (W_i(t) - \\ & - W_i(t - t_1) H(t - t_1)) \int_a^b x^p \Phi_i(x) dx \end{aligned} \quad (30)$$

where

$$U_i(t) = L^{-1} \left[\frac{(\tau_\alpha s^\alpha + 1)}{(\tau_\alpha s^{\alpha+1} + s + \kappa \lambda_i^2)} \frac{s\omega}{s^2 + \omega^2} \right] \quad (31)$$

$$V_i(t) = L^{-1} \left[\frac{\tau_\alpha s^\alpha + 1}{\tau_\alpha s^{\alpha+1} + s + \kappa \lambda_i^2} \right] \quad (32)$$

$$W_i(t) = L^{-1} \left[\frac{\tau_\alpha s^\alpha + 1}{s^2 (\tau_\alpha s^{\alpha+1} + s + \kappa \lambda_i^2)} \right], \quad (33)$$

The inverse of the Laplace transforms (31–33) can be numerically determined. For some values of the order α , an analytical form of these inverse transforms can be obtained. For $\alpha = 1$ (the hyperbolic heat transfer [19–20]), inverse transforms (31–33) can be presented in the form

$$\begin{aligned} U_i(t) = & \frac{\kappa \lambda_i^2 \omega}{\omega^2 + (\tau_\alpha \omega^2 - \kappa \lambda_i^2)^2} \\ & \left\{ \cos \omega t + \frac{\omega(1 + \tau_\alpha(\tau_\alpha \omega^2 - \kappa \lambda_i^2))}{\kappa \lambda_i^2} \sin \omega t - \right. \\ & \left. - e^{-\frac{t}{2\tau_\alpha}} \cdot \left[\frac{2\tau_\alpha(\tau_\alpha \omega^2 - \kappa \lambda_i^2)}{\Delta_i} \sinh \frac{t\Delta_i}{2\tau_\alpha} + P_i(t) \right] \right\} \end{aligned} \quad (34)$$

$$V_i(t) = e^{-\frac{t}{2\tau_\alpha}} P_i(t) \quad (35)$$

$$\begin{aligned} W_i(t) = & \frac{1}{\kappa^2 \lambda_i^4} \left[\kappa \lambda_i^2 (t + \tau_\alpha) - 1 - e^{-\frac{t}{2\tau_\alpha}} \cdot \right. \\ & \left. \cdot \left(\kappa \tau_\alpha \lambda_i^2 \left(\frac{3}{\Delta_i} \sinh \frac{t\Delta_i}{2\tau_\alpha} + \cosh \frac{t\Delta_i}{2\tau_\alpha} \right) - P_i(t) \right) \right] \end{aligned} \quad (36)$$

where $P_i(t) = \frac{1}{\Delta_i} \sinh \frac{t\Delta_i}{2\tau_\alpha} + \cosh \frac{t\Delta_i}{2\tau_\alpha}$, $\Delta_i = \sqrt{1 - 4\tau_\alpha \kappa \lambda_i^2}$. Hence, the analytical solution of the initial problem (22–24) for $\alpha = 1$ is given by equations (30, 34–36).

3.2. Solution of the eigenproblem in a rectangular coordinate system. The functions $\Phi_i(x)$ occurring in Eq. (30) are obtained by solving the eigenproblem (15–17). For the problem of the heat conduction in a slab ($p = 0$), without loss of generality, we assume that $a = 0$. In this case, the functions $\Phi_i(x)$ are

$$\Phi_i(x) = k \lambda_i \cos \lambda_i x + h_a \sin \lambda_i x \quad (37)$$

and the square norm of the i -th eigenfunction defined by equation (19), is given by

$$\begin{aligned} N_i = & \frac{1}{2} \left(b(h_a^2 + k^2 \lambda_i^2) + h_a k (1 - \cos(2b\lambda_i)) \right) + \\ & + \frac{1}{4\lambda_i} (-h_a^2 + k^2 \lambda_i^2) \sin(2b\lambda_i) \end{aligned} \quad (38)$$

where λ_i are roots of equation

$$(h_a + h_b) k \lambda \cos b\lambda - (k^2 \lambda^2 - h_a h_b) \sin b\lambda = 0. \quad (39)$$

We assume the initial distribution of temperature in the slab in the form

$$f(x) = \hat{T}_a \left(1 - \frac{x}{b} \right), \quad x \in [0, b]. \quad (40)$$

In this case, the function $F_i(t)$ defined by (23) for the problem of the heat conduction in the slab can be rewritten as

$$\begin{aligned} F_i(t) = & \frac{\kappa G_1}{N_i k} (t + \tau_\alpha - (t + \tau_\alpha - t_1) H(t - t_1)) \cdot \\ & \cdot \left(k \sin \lambda_i b + \frac{h_a}{\lambda_i} (1 - \cos \lambda_i b) \right) - \\ & - \frac{\hat{T}_a h_a \omega}{N_i \lambda_i} \left(\tau_\alpha t^{-\alpha} E_{\alpha, \beta}(-\omega^2 t^2) + \cos \omega t \right) \end{aligned} \quad (41)$$

where $E_{\alpha, \beta}$ is a two-parameter Mittag-Leffler function [18].

The solution of the eigenproblem (15–17) can be similarly derived for the heat conduction in a cylinder ($p = 1$) and in a sphere ($p = 2$).

4. Numerical examples

The solution in the Laplace transform domain, presented in the previous Section, will be used to investigate the effect of the fractional order of the Caputo derivative and the phase-lag parameter on the temperature distribution in a slab. The inverse Laplace transforms were numerically determined by using the Gaver formula [21]

$$f(t) \simeq n v \binom{2n}{n} \sum_{i=0}^n (-1)^i \binom{n}{i} \bar{f}((n+i)v) \quad (42)$$

where $v = (\ln 2)/t$ and n is a fixed positive integer number.

The numerical calculations were performed for the following data: the width of the slab is $b = 0.3$ m, the thermal diffusivity is $\kappa = 3.352 \cdot 10^{-6}$ m²/s, the thermal conductivity is $k = 16$ W/(m · K), the outer heat transfer coefficients are $h_a = h_b = 400$ W/(m² · K), $\hat{T}_a = 60^\circ\text{C}$, the parameters of the volumetric heat generation are $G_1 = 8.0 \cdot 10^4$ W/m³ and $t_1 = 600$ s. The computations were carried out by using the Mathematica package [22].

The non-dimensional temperatures $\hat{T} = T/\hat{T}_a$ computed by using the Gaver formula for numerical inversion of the Laplace transforms (NILT) and the results obtained by using the exact solution for $\alpha = 1.0$ and various values of $\hat{x} = x/b$ and $\hat{t} = \kappa t/b^2$ are presented in Table 1.

Table 1
The non-dimensional temperature $\hat{T}(\hat{x}, \hat{t})$ for $\alpha = 1.0$, computed by using the exact solution and by using numerical inversion of the Laplace transform (NILT)

\hat{x}	$\hat{t} = 0.1$		$\hat{t} = 0.5$		$\hat{t} = 1.0$	
	Exact	NILT	Exact	NILT	Exact	NILT
0	1.13109	1.13111	1.37309	1.37308	1.39375	1.39374
0.2	1.22487	1.22491	1.78681	1.78681	1.83467	1.83466
0.4	1.16470	1.16476	1.91273	1.91273	1.97608	1.97606
0.6	1.00168	1.00174	1.75484	1.75483	1.81818	1.81817
0.8	0.74068	0.74072	1.31313	1.31313	1.36098	1.36097
1.0	0.33571	0.33573	0.58361	0.58361	0.60428	0.60427

In calculations it was assumed that $\hat{\tau} = \tau\kappa/b^2 = 0.001$. The maximum of the relative errors $|Exact - NILT|/Exact$ does not exceed $6 \cdot 10^{-5}$. The good accordance of these numerical results obtained for $\alpha = 1.0$ allows one to use this method for numerical inversion of the Laplace transform for other values of the order α .

The non-dimensional temperature as a function α for different points of the slab \hat{x} and different values of dimensionless time \hat{t} are presented in Fig. 1. The calculations were performed assuming that the ambient temperature is constant $T_a = 60^\circ\text{C}$. Changes of temperature in the slab are caused by activity of the heat source which is specified by equation (29). It is observed that the temperature in the slab is higher for the lower orders of the Caputo derivative occurring in the heat conduction equation. The small differences of the temperatures occur for the order α belonging to interval $[0.5; 1]$.

The results of the numerical investigation of the effect of the oscillating ambient temperature on the temperature in the slab for different values of the order α are presented in Fig. 2. The calculations were performed for the slab without inner heat generation and with assumption that the changes of the ambient temperature at $x = 0$ according to equation (28) with $\omega = 0.001$ s⁻¹. The results for $\alpha = 0$ were obtained with assumption that $\hat{\tau} = 0$. Therefore, the curves for $\alpha = 0$ presented in Figs 2a-d correspond to the classical heat transfer model. For other cases of the order α , the non-dimensional phase-lag parameter was assumed that $\hat{\tau} = 0.001$. As is expected, the

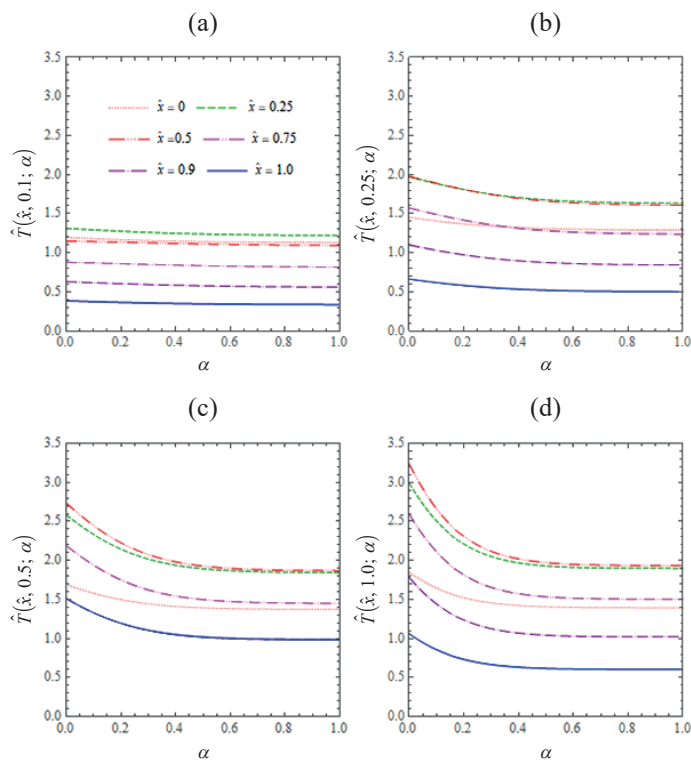


Fig. 1. The non-dimensional temperature $\hat{T}(\hat{x}, \hat{t}; \alpha)$ as a function of the variable α for various values of time \hat{t} : a) $\hat{t} = 0.1$; b) $\hat{t} = 0.25$; c) $\hat{t} = 0.5$; d) $\hat{t} = 1.0$

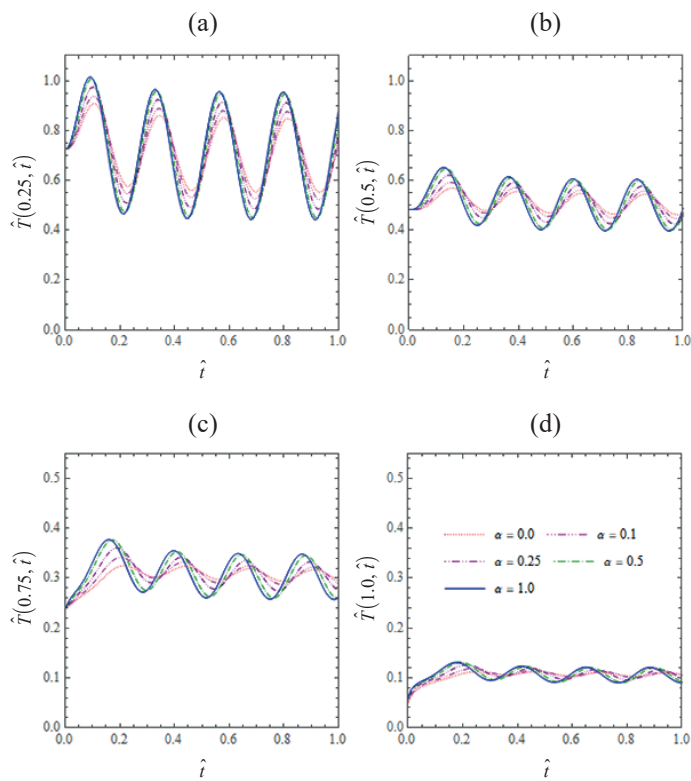


Fig. 2. The non-dimensional temperature $\hat{T}(\hat{x}, \hat{t})$ as a function of the variable \hat{t} for various values of α and the space variable \hat{x} : a) $\hat{x} = 0.25$; b) $\hat{x} = 0.5$; c) $\hat{x} = 0.75$; d) $\hat{x} = 1.0$

amplitudes of the temperature decrease with increasing of the distance from the heating boundary $x = 0$. The fractional derivative order occurring in the heat conduction model can be interpreted as a thermal damping coefficient which causes decreasing to the amplitude of the temperature changes. The numerical results showing the significance of the phase-lag parameter are assembled in Table 2. In fact, the value of this parameter will be chosen based on experimental data. Higher values of the parameter cause bigger lagging of the temperature growth in the considered region.

5. Conclusions

The solution of the fractional single-phase-lagging heat conduction problem in the form of the eigenfunctions series has been presented. The time-dependent coefficient of the series using the Laplace transform technique was determined by solving the time-fractional differential equation. Computational examples showed that the Gaver method can be used to the numerical inversion of the obtained Laplace transforms. It was found that significant influence on the temperature distribution in the slab has the time-fractional derivative (occurring in the heat conduction model) of the smaller order. It results in a decrease of amplitude of an oscillate temperature, i.e. it can be treated as a thermal damping coefficient. The higher phase-lag parameter

occurring in the mathematical model causes the higher delay of the change of the temperature. Parameters characterizing the heat conduction (order of the fractional derivative, relaxation time) should be chosen for better compatibility of the temperature distribution obtained using the fractional single phase-lag model and experimental data (calibration of the model [17]). Although the presented solution has been used for numerical investigation of the influences of the fractional derivative order and the phase-lag parameter on temperature distribution in the slab, it can be used to analysis of the fractional heat conduction in the hollow cylinder and the hollow sphere.

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Table 2
The non-dimensional temperature $\hat{T}(\hat{x}, \hat{t})$ for various values of the lag parameter \hat{t}

$\alpha = 0.1$									
\hat{t}	$\hat{x} = 0.25$			$\hat{x} = 0.5$			$\hat{x} = 0.75$		
	$\hat{t} = 10^{-5}$	$\hat{t} = 10^{-4}$	$\hat{t} = 10^{-3}$	$\hat{t} = 10^{-5}$	$\hat{t} = 10^{-4}$	$\hat{t} = 10^{-3}$	$\hat{t} = 10^{-5}$	$\hat{t} = 10^{-4}$	$\hat{t} = 10^{-3}$
0.1	0.9746	0.9627	0.9487	0.5744	0.5651	0.5553	0.3069	0.3028	0.2984
0.5	0.5857	0.5840	0.5834	0.4446	0.4534	0.4638	0.3070	0.3122	0.3170
1.0	0.7416	0.7281	0.7139	0.4466	0.4479	0.4506	0.2820	0.2870	0.2921
$\alpha = 0.5$									
\hat{t}	$\hat{x} = 0.25$			$\hat{x} = 0.5$			$\hat{x} = 0.75$		
	$\hat{t} = 10^{-5}$	$\hat{t} = 10^{-4}$	$\hat{t} = 10^{-3}$	$\hat{t} = 10^{-5}$	$\hat{t} = 10^{-4}$	$\hat{t} = 10^{-3}$	$\hat{t} = 10^{-5}$	$\hat{t} = 10^{-4}$	$\hat{t} = 10^{-3}$
0.1	1.0281	1.0269	1.0212	0.6261	0.6216	0.6071	0.3330	0.3300	0.3218
0.5	0.6057	0.5983	0.5768	0.4073	0.4066	0.4073	0.2732	0.2759	0.2859
1.0	0.8126	0.8046	0.7794	0.4540	0.4489	0.4354	0.2600	0.2595	0.2605
$\alpha = 1.0$									
\hat{t}	$\hat{x} = 0.25$			$\hat{x} = 0.5$			$\hat{x} = 0.75$		
	$\hat{t} = 10^{-5}$	$\hat{t} = 10^{-4}$	$\hat{t} = 10^{-3}$	$\hat{t} = 10^{-5}$	$\hat{t} = 10^{-4}$	$\hat{t} = 10^{-3}$	$\hat{t} = 10^{-5}$	$\hat{t} = 10^{-4}$	$\hat{t} = 10^{-3}$
0.1	1.0288	1.0289	1.0316	0.6282	0.6281	0.6274	0.3343	0.3342	0.3329
0.5	0.6094	0.6088	0.6037	0.4078	0.4075	0.4042	0.2719	0.2720	0.2717
1.0	0.8165	0.8162	0.8141	0.4564	0.4561	0.4524	0.2601	0.2601	0.2581

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