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Application of Duhamel's theorem in the thermal field analysis of a DC cable with two layers of insulation

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Abstract. In the paper we presented an asymptotic method of determination of the transient thermal field in a three-zone cable (i.e. in a core and in two layers of insulation). A basis of our method was decomposition of the system. We considered separately the region of the core, which we modelled by a concentrated element of the first order. In turn, we treated the two-zone region of insulation as the system of distributed parameters. In the analysis we applied Duhamel's theorem and the method of separation of variables. Eigenvalues of the boundary-initial problem and coefficients of eigenfunctions we determined numerically. In the result we obtained spatial-temporal heat-up curves of the investigated cable. Our results we presented in a graphic form.

Key words: analytical-numerical methods of the field theory, transient heat flow, current transmission.

1. Introduction

Nowadays transmission DC systems become more and more important. They have many advantages [1] (among others power losses and voltage drop in DC systems are smaller comparing with analogous AC systems) . In the Future subsequent increase of application of DC leads and cables is expected. Technological development will enable manufacture of conducting polymers and small size modules converting high voltage DC into a low one [2, 3]. For obvious reasons thermal processes dynamics has significant influence on the life time of current lines. Therefore the transient thermal field analysis in transmission DC systems has a great importance.

The paper refers to the recent publication [4]. In [4] the asymptotic method of analysis of the transient thermal field in a cable with single insulation layer was presented. The basis of considerations was decomposition of the system. In the present article the range of applications of the mentioned method was considerably extended. Namely, an additional (third) region of the cable was considered. This way a model of the system with two layers of insulation was developed. Cables of this type are characterized by better electrical and mechanical durability. In the consequences they are more resistant to external factors.

A cross section of the investigated cable is shown in Fig. 1 (without preserving proportions between the radii of particular layers). The internal region is a copper core (index 1). The middle zone is rubber insulation (index 2). The external coat consists of polyvinyl chloride PVC (index 3). The stated problem relies on determination of the spatial-temporal distribution of a temperature in the described cross section. Basic advantage of the proposed solution is the analytical form of results. They base on the system decomposition on a conducting and non-conducting part. Duhamel's theorem [5, 6] and the method of variables separation [7, 8] are applied in the analysis. However step characteristics of the system are not orthogonal functions in the rings of insulation. That is the basic mathematical difficulty which did not appear in [4]. For this reason some arguments of determined distributions are computed numerically.

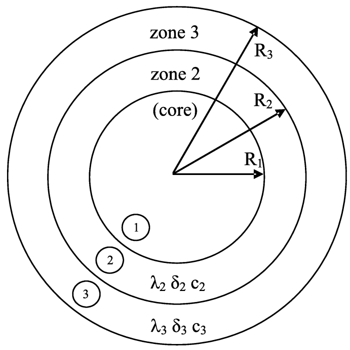

Fig. 1. Cross-section of the model of a cable with two layers of insulation

A bundle structure of the core significantly mitigate an eventual skin-effect for the AC flow. Lack of additional conducting elements excludes inducing eddy currents in the cable lamination. For this reasons the presented analysis can be referred to an AC current of the equivalent rms value.

2. Cable decomposition on the conducting and non-conducting part. The boundary-initial problem of the thermal field in insulations

The density of a direct current is the same in the whole cross section of a core. Besides its thermal conductivity is significantly larger than conductivity of insulating layers (in case of a copper, rubber and PVC over two thousand times). In the results it can be assumed, that after switch on of the pow-

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er supply the core is heating uniformly in its whole volume. It means that the selected conducting sub-region can be approximated by a concentrated inert element of the first order. Its step response is well known [9]

$$T_1(r,t) = T_1(t) = T_a + \nu_{ll}\left(1 - e^{-\frac{t}{\tau_c}}\right) \quad \text{for} \quad t > 0 \quad \text{and} \quad 0 \le r \le R_1, \tag{1}$$

where: $T_1(r,t)$ – temperature distribution in the core, r – radial coordinate, t – time, T_a – ambient temperature, $\nu_{ll} = T_{ll} - T_a$, T_{ll} – sustained admissible temperature of middle insulation, τ_c – time constant of the core.

A time constant of the core was determined similarly like in [10], but with a greater error. It results from the minimization of a measure of the temperature discontinuity on two boundaries $r = R_1$ and $r = R_2$ (in [10] only one boundary was discussed). Other estimations of τ_c were given in [4].

The second selected sub-region is the concentric system of rings: insulation ($R_1 \le r \le R_2$) and coat ($R_2 \le r \le R_3$), where R_i – radii: internal ($i = 1$) and exterior ($i = 2$) of the middle insulation and cable ($i = 3$). Heat flows from the core to ambient through the layers of insulation. In the considered sub-system spatial temperature changes cannot be neglected because of cooling. It leads to the model of distributed parameters. It is described by the boundary-initial problem, which is formulated with respect to thermal increases

$$v_i(r,t) = T_i(r,t) - T_a, \tag{2}$$

where: $T_i(r,t)$ – spatial-temporal distributions of a temperature, $i = 2$ (for insulation), $i = 3$ (for the coat). Temperature increases in the two-ring system are described by the equation of heat conduction [5, 11–13]

$$\frac{\partial^2 v_i(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial v_i(r,t)}{\partial r} - \frac{1}{\chi_i}\frac{\partial v_i(r,t)}{\partial t} = 0, \quad \text{for} \quad R_{i-1} \le r \le R_i, \quad t > 0 \quad \text{and} \quad i = 2,3, \tag{3a}$$

where: $\chi_i = \lambda_i/(c_i\delta_i)$ – diffusivity, λ_i – thermal conductivity, c_i – specific heat, δ_i – mass density, ($i = 2$ for insulation, $i = 3$ for the coat).

Before the power supply switch on the system was at the ambient temperature. Hence from (2) follow initial conditions

$$v_i(r,0) = 0 \quad \text{for} \quad R_{i-1} \le r \le R_i \quad \text{and} \quad i = 2,3. \tag{3b}$$

Because the core and insulation adhere tight to each other, on the boundary circle ($r = R_1$) increases have to be continuous. Taking advantage of (2) and (1) the first boundary condition was obtained

$$v_2(R_1,t) = \nu_{ll}\left(1 - e^{-\frac{t}{\tau_c}}\right) \quad \text{for} \quad t > 0. \tag{3c}$$

The outer surface of a cable is giving out the heat by convection and radiation. Hence follows the second boundary condition [14, 15]

$$\frac{\partial v_3(r,t)}{\partial r}\Big|_{r=R_3} = -\frac{\varepsilon}{\lambda_3} v_3(R_3,t) \quad \text{for} \quad t > 0, \tag{3d}$$

where ε denotes the total heat transfer coefficient.

Insulation and the coat of a cable are closely adherent to each other. Continuity conditions of the temperature increase and heat flux on the boundary of regions are then satisfied

$$v_2(R_2,t) = v_3(R_2,t), \tag{3e}$$

$$\lambda_2 \frac{\partial v_2(r,t)}{\partial r}\Big|_{r=R_2} = \lambda_3 \frac{\partial v_3(r,t)}{\partial r}\Big|_{r=R_2}, \tag{3f}$$

for $t \ge 0$. In order to solve the above boundary-initial problem (3a-f) Duhamel's theorem was utilised [5, 6]. It is particularly convenient for zero initial conditions (3b). Besides boundary conditions (3c,d) do not depend on spatial coordinates. In the result of that Duhamel's theorem simplifies to the following form

$$v_i(r,t) = \frac{1}{\nu_{ll}}\frac{\partial}{\partial t}\int_0^t h_i(r,\zeta)\, v_2(R_1, t-\zeta)\, d\zeta \quad \text{for} \quad R_{i-1} \le r \le R_i,\ t > 0 \quad \text{and} \quad i = 2,3, \tag{4}$$

where: $h_i(r,t)$ – step responses of the rings for excitation $\nu_{ll}\mathbf{1}(t)$ given on the boundary $r = R_1$ with zero initial conditions ($i = 2$ for insulation, $i = 3$ for the coat), $\mathbf{1}(t)$ – unit step function, ζ – dummy variable of integration. The right side of relation (4) was divided by amplification ν_{ll}. It was taken into account in step characteristics $h_i(r,t)$.

3. Step characteristics of non-conducting regions

3.1. Boundary-initial problem of step responses and their functions. It follows from (4) that condition for determination of the exponential response for excitation (3c) is knowledge of the fundamental solution (in this case step characteristics $h_i(r,t)$ for $i = 2,3$). They are described by the boundary-initial problem almost identical to (3a-f). Namely, in relations (3a,b,d,e,f) it is sufficient to exchange $\nu_i(r,t)$ by $h_i(r,t)$ (for $i = 2$ or $i = 3$), what is denoted by the symbol $v_i(r,t) \to h_i(r,t)$. Only the right side of (3c) changes in a significant way

$$h_2(R_1,t) = \nu_{ll} \quad \text{for} \quad t > 0. \tag{5}$$

Step responses were determined by superposition of states [11, 14]

$$h_i(r,t) = h_{is}(r) + h_{it}(r,t), \tag{6}$$

where: $i = 2$ or $i = 3$, $h_{is}(r)$ – stationary component $\left(\lim\limits_{t\to\infty} h_i(r,t) = h_{is}(r)\right)$, $h_{it}(r,t)$ – transient component $\left(\lim\limits_{t\to\infty} h_{it}(r,t) = 0\right)$.

A boundary problem for the stationary component was obtained by exchange of functions $v_i(r,t) \to h_{is}(r)$ in (3a,d,e,f) and $h_2(R_1,t) \to h_{2s}(R_1)$ in (5). In the result of that partial derivatives will change to the ordinary ones with respect to r, but become zero with respect to t. Then by virtue of the formula for the product derivative notation of the left side (3a) (for $i = 2,3$) was shortened. After double integration of the sides of the changed Eq. (3a) (with $i = 2,3$) and constants determination from modified conditions (3d,e,f), (5) it was finally obtained

$$h_{2s}(r) = \nu_{ll}\left(1 + \frac{1}{\dfrac{\lambda_2}{R_3\varepsilon} + \dfrac{\lambda_2}{\lambda_3}\ln\dfrac{R_3}{R_2} + \ln\dfrac{R_2}{R_1}}\ln\frac{R_1}{r}\right) \quad \text{for} \quad R_1 \le r \le R_2, \tag{7a}$$

$$h_{3s}(r) = \nu_{ll}\left(\frac{1}{\dfrac{\lambda_3}{R_3\varepsilon} + \ln\dfrac{R_3}{R_2} + \dfrac{\lambda_3}{\lambda_2}\ln\dfrac{R_2}{R_1}}\left(\ln\frac{R_3}{r} + \frac{\lambda_3}{R_3\varepsilon}\right)\right) \quad \text{for} \quad R_2 \le r \le R_3. \tag{7b}$$

In turn, the boundary-initial problem for transient components follows from (6). The problem of a step response and stationary component was utilized, as well. Finally it is sufficient to exchange in (3a,d,e,f) $v_i(r,t) \to h_{it}(r,t)$ (for $i = 2,3$). Only initial conditions (3b) will change

$$h_{it}(r,0) = -h_{is}(r) \quad \text{for} \quad R_{i-1} \le r \le R_i \quad \text{and} \quad i = 2,3, \tag{8a}$$

and the boundary one (5)

$$h_{2t}(R_1,t) = 0 \quad \text{for} \quad t > 0. \tag{8b}$$

The modified partial differential equations type (3a) were solved by the separation of variables [7, 8, 11]. It leads to determination of eigenfunctions: Bessel's $(J_0(z))$ and Neumann's $(Y_0(z))$ of zero order with respect to the radial coordinate and exponential with respect to the time. In addition in the region of a coating the number of constants was reduced taking advantage of Hankel's condition (3d) (after exchange $v_3(r,t) \to h_{3t}(r,t)$). Finally based on (6) we obtained

$$h_2(r,t) = h_{2s}(r) + \sum_{n=1}^{\infty}\left[P_n J_0\left(\alpha_n\frac{r}{R_1}\right) + V_n Y_0\left(\alpha_n\frac{r}{R_1}\right)\right]e^{-\chi_2\alpha_n^2\frac{t}{R_1^2}} \quad \text{for} \quad R_1 \le r \le R_2, \quad t > 0, \tag{9a}$$

$$h_3(r,t) = h_{3s}(r) + \sum_{n=1}^{\infty}\Gamma_n Z_0\left(\beta_n\frac{r}{R_1}\right)e^{-\chi_3\beta_n^2\frac{t}{R_1^2}} \quad \text{for} \quad R_2 \le r \le R_3 \quad \text{and} \quad t > 0, \tag{9b}$$

where

$$Z_k\left(\beta_n\frac{r}{R_1}\right) \overset{df}{=} J_k\left(\beta_n\frac{r}{R_1}\right)\left[Y_0\left(\beta_n\frac{R_3}{R_1}\right) - \eta\beta_n Y_1\left(\beta_n\frac{R_3}{R_1}\right)\right] - Y_k\left(\beta_n\frac{r}{R_1}\right)\left[J_0\left(\beta_n\frac{R_3}{R_1}\right) - \eta\beta_n J_1\left(\beta_n\frac{R_3}{R_1}\right)\right] \quad \text{for} \quad k = 0 \quad \text{or} \quad k = 1, \tag{9c}$$

(α_n, β_n) – eigenvalues of the boundary-initial problem of the step response for transient components respectively for $i = 2$ or $i = 3$ (i.e. constants of separation in particular zones multiplied by R_1), $\eta = \lambda_3/(R_1\varepsilon)$ – dimensionless constant, (P_n, V_n, Γ_n) – coefficients of the functions described the field in insulation (P_n, V_n) and in the outer coat of a cable (Γ_n), $(h_{2s}(r), h_{3s}(r))$ – stationary components of the step response given by formulas (7a,b).

The coefficients P_n, V_n, Γ_n and eigenvalues α_n, β_n appearing in (9a,b) have to be determined, as well.

3.2. Coefficients of functions of the step responses. Inequality to zero of the coefficients P_n, V_n, Γ_n is sufficient for the existence of transient components $h_{it}(r,t)$ in distributions (9a,b). In order to determine the existence condition of $h_{it}(r,t)$ (9a,b) was substituted to (3e,f) (after exchange $v_i(r,t) \to h_i(r,t)$ for $i = 2,3$) and boundary condition (8b) was utilized. This way a homogeneous system of three equations with respect to P_n, V_n, Γ_n was obtained. This system has non-trivial solutions (i.e. $P_n \ne 0$, $V_n \ne 0$, $\Gamma_n \ne 0$), when its main determinant is equal to zero

$$\begin{aligned}\Delta(\alpha_n,\beta_n) = {} & \beta_n Z_1\left(\beta_n\frac{R_2}{R_1}\right)\left(Y_0(\alpha_n)J_0\left(\alpha_n\frac{R_2}{R_1}\right) - J_0(\alpha_n)Y_0\left(\alpha_n\frac{R_2}{R_1}\right)\right) \\ & + \frac{\lambda_2}{\lambda_3}\alpha_n Z_0\left(\beta_n\frac{R_2}{R_1}\right)\left(J_0(\alpha_n)Y_1\left(\alpha_n\frac{R_2}{R_1}\right) - Y_0(\alpha_n)J_1\left(\alpha_n\frac{R_2}{R_1}\right)\right) = 0,\end{aligned} \tag{10}$$

where $Z_k\left(\beta_n\frac{R_2}{R_1}\right)$ is computed from (9c) introducing $r = R_2$ for $k = 0$ or $k = 1$.

Condition (10) is the equation of eigenvalues. It is satisfied by the two series of solutions: α_n, β_n. For this reason the sequences $\left\{P_n J_0\left(\alpha_n\frac{r}{R_1}\right) + V_n Y_0\left(\alpha_n\frac{r}{R_1}\right)\right\}$ in the range $\langle R_1, R_2\rangle$ and $\left\{Z_0\left(\beta_n\frac{r}{R_1}\right)\right\}$ in the range $\langle R_2, R_3\rangle$ do not form orthogonal systems of functions with the weight r. The coefficients P_n, V_n, Γ_n cannot be determined in a classic way on the basis of properties of generalized Fourier-Bessel series [7]. For that reason the summation of series (9a,b) was limited to a finite number L of terms.

In order to determine coefficients Γ_n the transient component of relation (9b) was substituted to (8a) for $i = 3$ with $r \in \langle R_2, R_3\rangle$

$$\sum_{n=1}^{L}\Gamma_n Z_0\left(\beta_n\frac{r}{R_1}\right) = -h_{3s}(r). \tag{11}$$

It follows from the above, that

$$\sum_{n=1}^{L}\Gamma_n\left[Z_0\left(\beta_n\frac{r}{R_1}\right), rZ_0\left(\beta_k\frac{r}{R_1}\right)\right] = -\left[h_{3s}(r), rZ_0\left(\beta_k\frac{r}{R_1}\right)\right], \tag{12a}$$

where the following scalar products [7, 8] were introduced with the weight r in the interval $\langle R_2, R_3\rangle$

$$\Omega_{nk} = \left[Z_0\left(\beta_n \frac{r}{R_1}\right), rZ_0\left(\beta_k \frac{r}{R_1}\right)\right] = \int_{R_2}^{R_3} rZ_0\left(\beta_n \frac{r}{R_1}\right) Z_0\left(\beta_k \frac{r}{R_1}\right) dr$$

$$= \begin{cases} R_1R_2 \frac{\beta_n Z_1\left(\beta_n \frac{R_2}{R_1}\right) Z_0\left(\beta_k \frac{R_2}{R_1}\right) - \beta_k Z_0\left(\beta_n \frac{R_2}{R_1}\right) Z_1\left(\beta_k \frac{R_2}{R_1}\right)}{\beta_k^2 - \beta_n^2} & \text{for} \quad n \neq k, \\ \frac{1}{2}R_3^2 \left\{\left[Z_0\left(\beta_k \frac{R_3}{R_1}\right)\right]^2 + \left[Z_1\left(\beta_k \frac{R_3}{R_1}\right)\right]^2\right\} - \frac{1}{2}R_2^2 \left\{\left[Z_0\left(\beta_k \frac{R_2}{R_1}\right)\right]^2 + \left[Z_1\left(\beta_k \frac{R_2}{R_1}\right)\right]^2\right\} & \text{for} \quad n = k, \end{cases} \tag{12b}$$

$$\Lambda_k = -\left[h_{3s}(r), rZ_0\left(\beta_k \frac{r}{R_1}\right)\right] = -\int_{R_2}^{R_3} rh_{3s}(r) Z_0\left(\beta_k \frac{r}{R_1}\right) dr$$

$$= \frac{-v_{ll}}{\frac{\lambda_3}{\varepsilon R_3} + \ln\frac{R_3}{R_2} + \frac{\lambda_3}{\lambda_2}\ln\frac{R_2}{R_1}} \left[\frac{\lambda_3 R_1}{\varepsilon \beta_k} Z_1\left(\beta_k \frac{R_3}{R_1}\right) + \frac{R_1^2}{\beta_k^2}\left(Z_0\left(\beta_k \frac{R_2}{R_1}\right) - Z_0\left(\beta_k \frac{R_3}{R_1}\right)\right) - \frac{R_2R_1}{\beta_k} Z_1\left(\beta_k \frac{R_2}{R_1}\right)\left(\ln\frac{R_3}{R_2} + \frac{\lambda_3}{\varepsilon R_3}\right)\right]. \tag{12c}$$

Then coefficients Γ_n can be determined from the following system of equations

$$\sum_{n=1}^{L} \Omega_{nk}\Gamma_n = \Lambda_k \quad \text{for} \quad k = 1, 2, 3, ...L. \tag{13}$$

In turn, to determine coefficients P_n, V_n the transient component of relation (9a) was substituted to (8a) for $i = 2$ with $r \in \langle R_1, R_2\rangle$. Then a set of scalar products with the weight r in the given interval was formed

$$\begin{cases} \sum_{n=1}^{L}\left[P_n J_0\left(\alpha_n \frac{r}{R_1}\right) + V_n Y_0\left(\alpha_n \frac{r}{R_1}\right), rJ_0\left(\alpha_k \frac{r}{R_1}\right)\right] = -\left[h_{2s}(r), rJ_0\left(\alpha_k \frac{r}{R_1}\right)\right], \\ \sum_{n=1}^{L}\left[P_n J_0\left(\alpha_n \frac{r}{R_1}\right) + V_n Y_0\left(\alpha_n \frac{r}{R_1}\right), rY_0\left(\alpha_k \frac{r}{R_1}\right)\right] = -\left[h_{2s}(r), rY_0\left(\alpha_k \frac{r}{R_1}\right)\right]. \end{cases} \tag{14}$$

In order to simplify notation (14) the following denotations were introduced

$$\Phi_{nk} = \int_{R_1}^{R_2} rJ_0\left(\alpha_n \frac{r}{R_1}\right) J_0\left(\alpha_k \frac{r}{R_1}\right) dr$$

$$= \begin{cases} R_1R_2 \frac{\alpha_n J_1\left(\alpha_n \frac{R_2}{R_1}\right) J_0\left(\alpha_k \frac{R_2}{R_1}\right) - \alpha_k J_1\left(\alpha_k \frac{R_2}{R_1}\right) J_0\left(\alpha_n \frac{R_2}{R_1}\right)}{\alpha_n^2 - \alpha_k^2} - \mathrm{R}_1^2 \frac{\alpha_n J_1(\alpha_n) J_0(\alpha_k) - \alpha_k J_1(\alpha_k) J_0(\alpha_n)}{\alpha_n^2 - \alpha_k^2} & \text{for} \quad n \neq k \\ \frac{1}{2}R_2^2\left[J_0^2\left(\alpha_k \frac{R_2}{R_1}\right) + J_1^2\left(\alpha_k \frac{R_2}{R_1}\right)\right] - \frac{1}{2}R_1^2\left[J_0^2(\alpha_k) + J_1^2(\alpha_k)\right] & \text{for} \quad n = k, \end{cases} \tag{15a}$$

$$\Psi_{nk} = \int_{R_1}^{R_2} rY_0\left(\alpha_n \frac{r}{R_1}\right) J_0\left(\alpha_k \frac{r}{R_1}\right) dr$$

$$= \begin{cases} R_1R_2 \frac{\alpha_n J_0\left(\alpha_k \frac{R_2}{R_1}\right) Y_1\left(\alpha_n \frac{R_2}{R_1}\right) - \alpha_k J_1\left(\alpha_k \frac{R_2}{R_1}\right) Y_0\left(\alpha_n \frac{R_2}{R_1}\right)}{\alpha_n^2 - \alpha_k^2} - \mathrm{R}_1^2 \frac{\alpha_n Y_1(\alpha_n) J_0(\alpha_k) - \alpha_k J_1(\alpha_k) Y_0(\alpha_n)}{\alpha_n^2 - \alpha_k^2} & \text{for} \quad n \neq k \\ \frac{\mathrm{G}_{2,4}^{2,1}\left(\alpha_k \frac{R_2}{R_1}, \frac{1}{2} \middle| \begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 0, 0, -1, -\frac{1}{2} \end{matrix}\right)\mathrm{R}_2^2 - \mathrm{G}_{2,4}^{2,1}\left(\alpha_k, \frac{1}{2} \middle| \begin{matrix} \frac{1}{2}, -\frac{1}{2} \\ 0, 0, -1, -\frac{1}{2} \end{matrix}\right)\mathrm{R}_1^2}{2\sqrt{\pi}} & \text{for} \quad n = k, \end{cases} \tag{15b}$$

$\mathrm{G}_{\mathrm{p,q}}^{\mathrm{m,n}}\left(x, r \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right)$ – G Meijer's function [16, 17],

$$W_k = -\int_{R_1}^{R_2} rh_{2s}(r) J_0\left(\alpha_k \frac{r}{R_1}\right) dr = -v_{ll}\frac{R_1R_2}{\alpha_k} J_1\left(\alpha_k \frac{R_2}{R_1}\right) + v_{ll}\frac{R_1^2}{\alpha_k} J_1(\alpha_k)$$

$$- \frac{v_{ll}}{\frac{\lambda_2}{\varepsilon R_3} + \frac{\lambda_2}{\lambda_3}\ln\frac{R_3}{R_2} + \ln\frac{R_2}{R_1}} \left(\frac{R_1R_2}{\alpha_k}\ln\frac{R_1}{R_2} J_1\left(\alpha_k \frac{R_2}{R_1}\right) - \frac{R_1^2}{\alpha_k^2} J_0\left(\alpha_k \frac{R_2}{R_1}\right) + \frac{R_1^2}{\alpha_k} J_0(\alpha_k)\right). \tag{15c}$$

Scalar products in the bottom line of (14) were computed in the same way. In the result (14) transforms into the system of 2L independent equations, from which P_n, V_n can be determined

$$\begin{cases} \sum_{n=1}^{L}\left[P_n \Phi_{nk} + V_n \Psi_{nk}\right] = W_k & \text{for} \quad k = 1, 2, 3, ...L, \\ \sum_{n=1}^{L}\left[P_n \mathrm{O}_{nk} + V_n U_{nk}\right] = Q_k & \text{for} \quad k = 1, 2, 3, ...L. \end{cases} \tag{16}$$

Scalar products O_{nk}, U_{nk}, Q_k occurring in the bottom line of the system (16) are determined by similar relations as (15a), (15b), (15c), adequately. Namely, in order to obtain U_{nk} and Q_k it is necessary to exchange $J_k(...)$ by $Y_k(...)$ in (15a) and (15c) with preserve of the same arguments and orders of cylindrical functions. To obtain O_{nk} it is necessary to exchange in (15b) for $n \neq k$ function $J_k(...)$ by $Y_k(...)$ and vice versa for unchanged arguments and orders of the functions. The scalar product O_{nk} for $n = k$ is of the identical form as (15b) for $n = k$.

In computations of (12b,c), (15a,b,c) there were utilized:

– integrals of cylindrical functions given in [16],
– relation following from formula (16),
– integration method by the change of a variable and by parts,
– Meijer's functions G (15b) for $n = k$, which are defined by means of curvilinear integrals [16, 17].

The systems of linear Eqs. (13) and (16) enable to determine investigated coefficients P_n, V_n, Γ_n of the functions of characteristics. However prior knowledge of eigenvalues α_n, β_n of the step responses is required.

3.3. Eigenvalues of the step responses. Two series of eigenvalues α_n, β_n cannot be determined from one Eq. (10) (i.e. from the existence condition of transient components). In connection with that, (10) was supplemented by the continuity condition of the step characteristics on the boundary of regions (i.e. (3e) after exchange $v_i(r,t) \rightarrow h_i(r,t)$). Conjunction (10) and modified (3e) is a basis of the author's algorithm for determination of eigenvalues. Their consecutive pairs are computed in the following after each other blocks (Fig. 2). First a starting range of the investigated α_1, β_1 is determined. For this purpose a rough measure of discontinuity of the step characteristics on the circle $r = R_2$ is minimised. In i-th iteration the mentioned measure is expressed as

$$w_i(\beta_{1i}) = \underset{t}{Max} \left| h_3(R_2, t, \beta_{1i}) - h_2(R_2, t, \alpha_{1i}) \right|, \qquad (17)$$

where index i sufficiently increases a value of β_{1i}. Whereas α_{1i} depends on β_{1i} by virtue of Eq. (10). The right end of the investigated sector is found, when the value of (17) stops to decrease (left end is zero). Then eigenvalues α_1, β_1 are determined. They are found by reduction of the starting sector. For this purpose elements of the method of golden cut were utilised [18]. Computed this way points β_1', β_1'' of a cut are substituted to equation of eigenvalues (10). After its solution one obtains α_1', α_1''. These are the first terms of proper series. Then two functions were defined

$$f_1(t, \beta_1') = h_3(R_2, t, \beta_1') - h_2(R_2, t, \alpha_1'), \qquad (18a)$$

$$f_2(t, \beta_1'') = h_3(R_2, t, \beta_1'') - h_2(R_2, t, \alpha_1''). \qquad (18b)$$

On this stage the characteristics h_2, h_3 take into account the steady component and the first term of a transient component. Functions (18a,b) change signs with the time passing by and they describe temperature differences on the boundary of regions for $r = R_2$. In the continuation the sums of modules of the global maximum and minimum of functions (18a,b) are formed with respect to time (which informs a subscript t)

$$g(\beta_1') = \left| \underset{t}{Max} f_1(t, \beta_1') \right| + \left| \underset{t}{Min} f_1(t, \beta_1') \right|, \qquad (19a)$$

$$g(\beta_1'') = \left| \underset{t}{Max} f_2(t, \beta_1'') \right| + \left| \underset{t}{Min} f_2(t, \beta_1'') \right|. \qquad (19b)$$

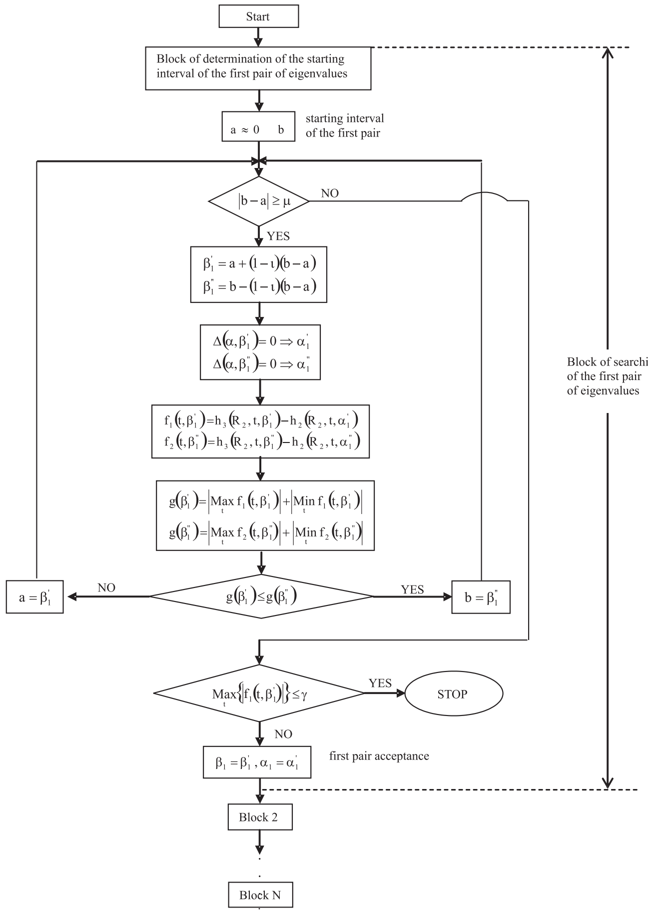

Fig. 2. Block diagram of algorithm for determination of eigenvalues

Formulas (19a,b) can be interpreted as a sum of the biggest deviations of functions (18a,b) with respect to the level zero (for which ideal continuity of the field for $r = R_2$ is fulfilled). Hence the continuity condition of step characteristics for $r = R_2$ is fulfilled with maximal accuracy for the minimum of function g. In accordance with the method of golden cut [18], the minimum is searched in the reduced interval, respectively. The operation is controlled by inequality $g(\beta_1') \leq g(\beta_1'')$ (Fig. 2). When a width of the interval reaches value less than μ, the first pair of eigenvalues is determined. If condition $\underset{t}{Max} \{|f_1(t, \beta_1')|\} \leq \gamma$ is satisfied, then the algorithm is terminated. Otherwise investigation of the next pair of eigenvalues begins. Executed operations are almost analogical as the one described in the above. The most important difference relies on taking into account of consecutive terms in relations (9a,b) (i.e. more accurate description of transient components). Obviously prior determined eigenvalues are the left ends of the next starting intervals.

4. Thermal field of non-conducting regions

Taking into account formula (2), the ambient temperature should be added to distributions (9a,b)

$$H_i(r,t) = T_a + h_i(r,t) \quad \text{for} \quad R_{i-1} \leq r \leq R_i, \quad t > 0 \quad \text{and} \quad i = 2, 3. \tag{20}$$

Then in relations (3c), (9a,b) variables were changed ($t \to t - \zeta$, $t \to \zeta$). Such a modified (3c), (9a,b) were substituted to (4). After integration with respect to ζ, differentiation with respect to t, arrangement and consideration of (2), investigated distributions of the temperature in the system were determined

$$T_2(r,t) = T_a + h_{2s}(r)\left(1 - e^{-\frac{t}{\tau_c}}\right) + \sum_{n=1}^{L} \frac{R_1^2}{(R_1^2 - \chi_2\alpha_n^2\tau_c)} \left[P_n J_0\left(\alpha_n \frac{r}{R_1}\right) + V_n Y_0\left(\alpha_n \frac{r}{R_1}\right)\right]\left(e^{-\chi_2\alpha_n^2\frac{t}{R_1^2}} - e^{-\frac{t}{\tau_c}}\right) \quad \text{for} \quad R_1 \leq r \leq R_2, \quad t > 0, \tag{21a}$$

$$T_3(r,t) = T_a + h_{3s}(r)\left(1 - e^{-\frac{t}{\tau_c}}\right) + \sum_{n=1}^{L} \frac{R_1^2\Gamma_n Z_0\left(\beta_n \frac{r}{R_1}\right)}{(R_1^2 - \chi_3\beta_n^2\tau_c)} \left(e^{-\chi_3\beta_n^2\frac{t}{R_1^2}} - e^{-\frac{t}{\tau_c}}\right) \quad \text{for} \quad R_2 \leq r \leq R_3, \quad t > 0. \tag{21b}$$

It is easy to notice, that relations (21a,b) for $t \to \infty$ describe a stationary distribution of the temperature $T_i(r,t) = T_a + h_{is}(r)$ (for $i = 2$ or $i = 3$). From another passage to the limit it follows that for $\tau_c \to 0$ formulas (21a,b) are identical with (20) (also for $i = 2, 3$).

5. Computational examples

On the basis of the presented method a computer program was developed in the Mathematica 6.0 environment [19]. The program computes: eigenvalues of the step responses, coefficients of the field functions, spatial-temporal distributions of the temperature (20), (21a,b) and visualises the results. The application was run on the laptop type computer Hewlet Packard ZE 5165 with a Pentium IV/2GHz processor.

The following data were assumed:

$$\begin{aligned}
&R_2 = 0.0155 \text{ m}; && c_2 = 1382 \text{ J}/(kgK);\\
&\delta_2 = 1200 \text{ kg/m}^3; && \lambda_2 = 0.163 \text{ W/(mK)};\\
&R_3 = 0.0173 \text{ m}; && c_3 = 1740 \text{ J}/(kgK);\\
&\delta_3 = 1350 \text{ kg/m}^3; && \lambda_3 = 0.17 \text{ W/(mK)};\\
&R_1 = 0.0117 \text{ m}; && \tau_c = 2150 \text{ s};\\
&T_{ll} = 70°\text{C}; && T_a = 25°\text{C};\\
&\varepsilon = 12.86 \text{ W/(m}^2\text{K)}; && \iota = \left(\sqrt{5} - 1\right)/2;\\
&\mu = 10^{-5}; && \nu_{ll} = 45°\text{C};\\
&\gamma = 0.05°\text{C}.
\end{aligned} \tag{22}$$

Results were presented in a graphic form. In Fig. 3 step characteristics (20) (for $i = 2, 3$) and the investigated temperature profiles (21a,b) at the selected points of a cable were drawn. In turn in Fig. 4a,b deviation τ_c influence on heat-up curves of middle circles: internal insulation ($r/R_1 = 1.162$) and the coating ($r/R_1 = 1.402$) were presented.

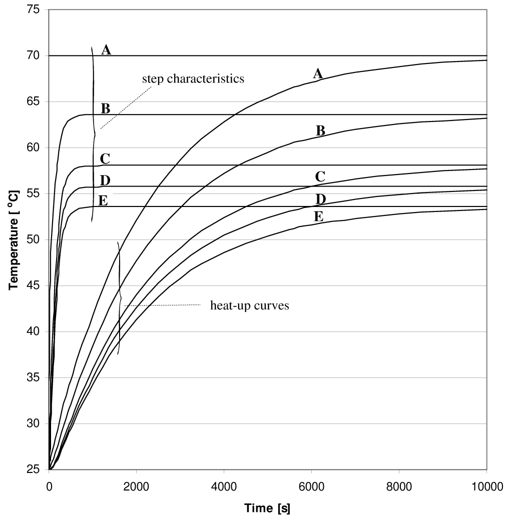

Fig. 3. Step characteristics (20) for $i = 2, 3$ and heat-up curves (21a,b) of the cable at the selected points of the cross-section: $A\,[0 \leq (r/R_1) \leq 1]$ – core; $B\,[(r/R_1) = 1.162]$ – middle of internal insulation; $C\,[(r/R_1) = 1.325]$ – boundary of insulation and a coating; $D\,[(r/R_1) = 1.402]$ – centre of a coating; $E\,[(r/R_1) = 1.479]$ – cable surface

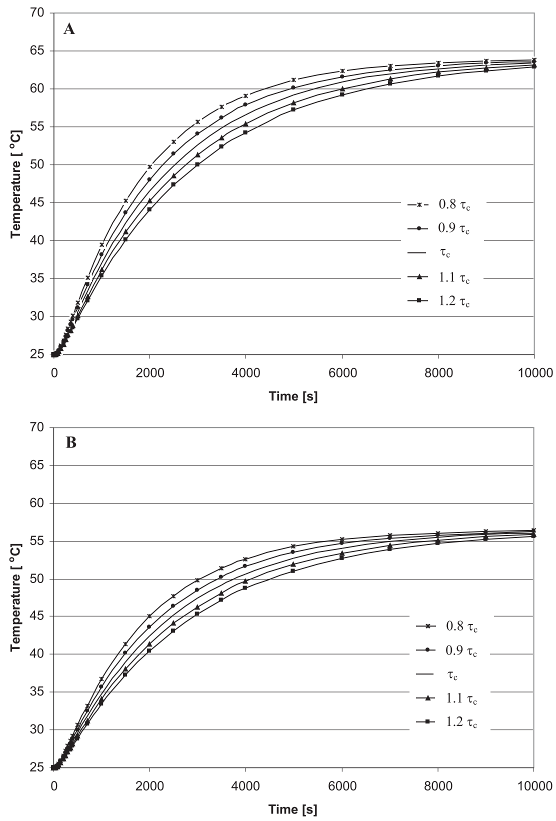

Fig. 4. Influence of a change of the core time constant on heat-up curves (21a,b) middle layers of insulation and the coating of a cable: $A\,[(r/R_1) = 1.162]$; $A\,[(r/R_1) = 1.402]$

Verification of the developed method was carried, as well. For this purpose obtained results were compared with computations by means of a finite element method (FE) [20]. The cross section from Fig. 1 was approximated by rectangular axial-symmetrical elements of a different size. Required data were taken from the set (22). They were supplemented by parameters of copper: $c_1 = 400$ J/(kgK); $\delta_1 = 8700$ kg/m^3; $\lambda_1 = 360$ W/(mK); $\rho = 1.94 \cdot 10^{-8}\Omega$m. The above symbols denote successively: specific heat, density, thermal conductivity and average resistivity in the range $< T_a,\ T_{ll} >$. The coefficient of strand packing is equal $(S_r/S_m) = 0.7$, where S_r and $S_m = \pi R_1^2$ mean its real and model cross section, respectively. In Appendix the steady state current rating is determined (for the given admissible sustained temperature). By means of (A6) and (22) the current load $I_{ll} = 666.47$ A was computed.

Verification was carried out taking advantage of the program NISA III/Heat Transfer [21]. In the result relative differences of heat-up curves

$$100\% \frac{[T_{FE}(r,t) - T_D(r,t)]}{T_{FE}(r,t)} \tag{23}$$

at the selected points of a cable are presented in Fig. 5. In formula (23) $T_D(r,t)$ means spatial-temporal distribution of the temperature obtained on the basis of Duhamel's theorem. The logarithmic scale was applied on the abscissa axis in Fig. 5 for more convenient observations (23) at the beginning of a transient state.

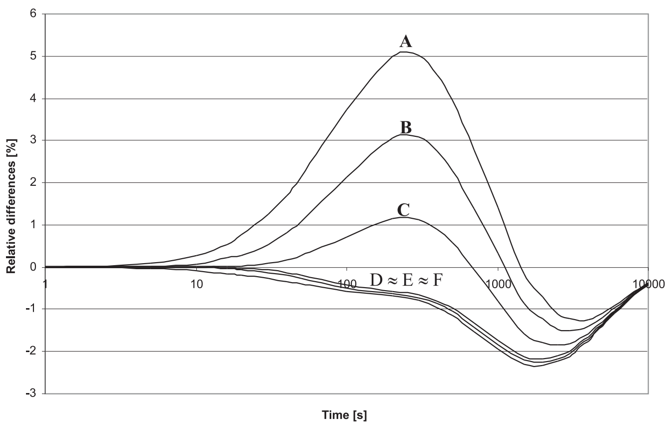

Fig. 5. Relative differences of heat-up curves determined by the finite element method (FE) and Duhamel's one (D) at the selected points of a cable: $A\,[(r/R_1) = 1.081]$; $B\,[(r/R_1) = 1.162]$ – centre of internal insulation; $C\,[(r/R_1) = 1.244]$; $D\,[(r/R_1) = 1.325]$ – boundary of insulation and the coating; $E\,[(r/R_1) = 1.402]$ – centre of the coating; $F\,[(r/R_1) = 1.479]$ – cable surface

6. Final remarks

From the considerations carried out result the following remarks:

A) The presented method considerably simplifies the transient thermal field analysis of a cable (Fig. 1). Because it was proved, that the three-zone system can be substituted by a two-zone one. In the result only two series of eigenvalues are determined instead of three, what significantly simplifies the algorithm of their computation.

B) The obtained results have a good physical interpretation. It follows from Fig. 3, that in both layers of insulation $(1 < (r/R_1) < (R_3/R_1))$ together with increase of the radial co-ordinate the value and velocity of rising of heat-up curves decreases. Physical reason of the above effect is growing distance of the point of observations from the heat source (a core). At the same time the distance from the cooled surface R_3/R_1 is reduced.

The heat-up curves of insulation increase considerably more slowly than their step characteristics (Fig. 3). Then despite very good propagation of heat in copper, condition (3c) cannot be substituted by a step increase of the temperature.

C) Inaccuracy in determination of the time constant τ_c of a core has some importance. As it follows from Fig. 4a,b, its underestimation increases the velocity of rising of heat-up curves and this way it reduces duration of the transient in relation to the accurate profile. In turn overestimation of τ_c causes opposite effects. Then deviation of τ_c deteriorates an accuracy of the presented method.

D) Relative differences (23) of heat-up curves computed by the method of finite elements (FE) and Duhamel's one (D) are the biggest in internal insulation (Fig. 5, curves a,b). The maximum is circa 5.1% and it is obtained after ca 600 s. In another points differences are smaller. The developed method should be admitted as verified.

Appendix

In the first turn the model was considered, in which a total filling of the cross section $S_m = \pi R_1^2$ by a copper core was assumed. The current rating is a such value of the current, which causes heat-up of the warmest points of insulation $(r = R_1)$ to the admissible sustained temperature T_{ll}. In the steady state the temperature decrease $T_{ll} - T_a$ can be connected with a generated power $I_m^2 \rho\lambda/S_m$ by thermal Ohm's law

$$T_{ll} - T_a = (R_{h2} + R_{h3} + R_a) I_m^2 \rho \frac{\lambda}{\pi R_1^2}, \tag{A1}$$

where:
λ – length of a cable section,

$$R_{h2} = \frac{1}{2\pi\lambda_2\lambda} \ln \frac{R_2}{R_1} \text{ – thermal resistance of insulation,} \tag{A2}$$

$$R_{h3} = \frac{1}{2\pi\lambda_3\lambda} \ln \frac{R_3}{R_2} \text{ – thermal resistance of a coat,} \tag{A3}$$

$$R_a = \frac{1}{2\pi R_3 \varepsilon \lambda} \text{ – thermal resistance of air.} \tag{A4}$$

In reality the cross section of a core (S_r) is smaller than $S_m = \pi R_1^2$. Thermal effects will be the same, if the current density in a real system and in the modelled one is the same

$$I_{ll} = \frac{S_r}{\pi R_1^2} I_m. \tag{A5}$$

After substitution of (A2)–(A4) to (A1) I_m was determined. After taking advantage of (A5) it was finally obtained

$$I_{ll} = \frac{S_r}{R_1}\sqrt{\frac{2\lambda_2\lambda_3\varepsilon R_3\left(T_{ll}-T_a\right)}{\rho\left[\lambda_3\varepsilon R_3\ln\left(R_2/R_1\right)+\lambda_2\varepsilon R_3\ln\left(R_3/R_2\right)+\lambda_2\lambda_3\right]}}. \tag{A6}$$

With the aid of the above formula a value of the current was computed, which was needed for verification of the earlier presented method.

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